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Towards a probability theory for product logic: States, integral representation and reasoning

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ABSTRACT

The aim of this paper is to extend probability theory from the classical to the product t-norm fuzzy logic setting. More precisely, we axiomatize a generalized notion of finitely additive probability for product logic formulas, called state, and show that every state is the Lebesgue integral with respect to a unique regular Borel probability measure. Furthermore, the relation between states and measures is shown to be one–one. In addition, we study geometrical properties of the convex set of states and show that extremal states, i.e., the extremal points of the state space, are the same as the truth-value assignments of the logic. Finally, we axiomatize a two-tiered modal logic for probabilistic reasoning on product logic events and prove soundness and completeness with respect to probabilistic spaces, where the algebra is a free product algebra and the measure is a state in the above sense.

1. Introduction

In his monograph [21], Hájek established the theoretical ground for a wide family of fuzzy (thus, many-valued) logics which, since then, has been significantly developed and further generalized, giving rise to a discipline that has been named Mathematical Fuzzy logic (MFL). Hájek's approach consists in fixing the real unit interval as standard domain to evaluate atomic formulas, while the evaluation of compound sentences only depends on the chosen operation which provides the semantics for the so called *strong conjunction* connective. His general approach to fuzzy logics is grounded on the observation that, if strong conjunction is interpreted by a continuous t-norm [22], then any other connective of a logic has a natural standard interpretation.

Among continuous t-norms, the so called Łukasiewicz, Gödel and product t-norms play a fundamental role. Indeed, Mostert–Shields' theorem [22] shows that a t-norm is continuous if and only if it can be built from the previous three ones by the construction of ordinal sum. In other words, a t-norm is continuous if and only if it is an ordinal sum of Łukasiewicz, Gödel and product t-norms. These three operations determine three different algebraizable propositional logics (bringing the same names as their associated t-norms), whose equivalent algebraic semantics are the varieties of MV, Gödel and product algebras respectively.

The first generalization of probability theory to the nonclassical settings of t-norm based fuzzy logics in Hájek sense, is due to Mundici who, in 1995, introduced the notion of *state* for the class of MV-algebras—the algebraic counterpart Łukasiewicz logic—with the aim of capturing the notion of average degree of truth of a proposition, [28].

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In that paper, states are functions mapping an MV-algebra to the real unit interval $[0,1]$, satisfying a normalization condition and the finite additivity law. Such functions suitably generalize the classical notion of finitely additive probability measures on Boolean algebras, in addition to corresponding to convex combinations of valuations of Łukasiewicz propositional logic. However, states and probability measures were previously studied in [9] (see also [10,30]) on Łukasiewicz tribes (σ-complete MV-algebras of fuzzy sets) as well as on other t-norm based tribes with continuous operations.

MV-algebraic states have been deeply studied in recent years, as they enjoy several important properties and characterizations (see [18] for a survey). One of the most important results in that framework is Kroupa–Panti theorem [29, §10], a representation result showing that every state of an MV-algebra is the Lebesgue integral with respect to a regular Borel probability measure. Moreover, the correspondence between states and regular Borel probability measures is one–one.

Many attempts of defining suitable notions of state in different structures have been made (see for instance [18, §8] for a short survey). In particular, in [5], the authors provide a definition of state for the Lindenbaum algebra of Gödel logic that corresponds to the integration of the n-place truth-functions corresponding to Gödel formulas, with respect to Borel probability measures on the real unit cube $[0,1]^n$. Moreover, such states are shown to correspond to convex combinations of finitely many truth-value assignments. Similar results have been obtained for the case Gödel logic expanded with Baaz–Monteiro operator Δ [1], and for the case of Nilpotent Minimum logic [4].

The aims of this contribution are the following: (1) we will introduce and study states for product logic—the remaining fundamental many-valued logic for which such a notion is still lacking—(2) we will prove that our axiomatization results in characterizing Lebesgue integrals of truth-functions of product logic formulas with respect to regular Borel probability measures, and (3) following similar lines to those of [14,20], we will axiomatize a modal expansion of Łukasiewicz logic for probabilistic reasoning on events described by formulas of product logic. In more detail, we show that states of the Lindenbaum algebra of product logic over n variables, i.e. the free n-generated product algebra, correspond, one–one, to regular Borel probability measures on $[0,1]^n$.[1] Moreover, and quite surprisingly since in the axiomatization of states the product t-norm operation is only indirectly involved via a condition concerning double negation, we prove that every state belongs to the convex closure of product logic valuations. Finally, these results will allow us to introduce a suitable class of probabilistic-like models with respect to which the modal logic we will introduce in Section 6 turns out to be sound and complete.

The paper is structured as follows. After this introduction, in Section 2 we will recall the functional representation of the free n-generated product algebra $\mathcal{F}_{\mathbb{P}}(n)$, as presented in [3] (see also [11]). We will easily prove that such functions, although they are not continuous, are indeed Borel measurable. In particular, from that functional representation of product logic functions, it follows that the domain $[0,1]^n$ of each such a function can be partitioned in locally compact and Hausdorff subsets of $[0,1]^n$, named G_ε (with ε varying in a certain set Σ, depending on the atoms of the Boolean skeleton of $\mathcal{F}_{\mathbb{P}}(n)$). More precisely, each G_ε is an F_σ set since, in fact, it is a countable union of a family $\{G^q_\varepsilon\}_{q\in(0,1]\cap\mathbb{Q}}$ of nested compact subsets of $[0,1]^n$, and hence it is a σ-locally compact set (see [33, §1.11]). Over each G_ε, the function is actually continuous. Moreover, any continuous function with domain one of the compact sets G^q_ε can be uniformly approximated by linear combinations of the functions of $\mathcal{F}_{\mathbb{P}}(n)$ restricted to such subsets.

In Section 3, we will axiomatize our notion of state of $\mathcal{F}_{\mathbb{P}}(n)$, and show its properties together with some examples. In particular, we will investigate states of the 1-generated free product algebra, and see how this analysis reflects into its spectral space.

In Section 4 we will prove our main result, that is to say, for every state s of $\mathcal{F}_{\mathbb{P}}(n)$ there is a unique Borel probability measure μ on $[0,1]^n$ such that s is the Lebesgue integral with respect to μ, and vice versa, every such an integral operator is a state in our sense. In Section 5, we shall prove that the state space of $\mathcal{F}_{\mathbb{P}}(n)$ is convex and closed. Thus, via Krein–Milman theorem (see for instance [19]) every state is a convex combination of extremal ones. We will hence characterize the extremal states, proving that they coincide with the homomorphisms of $\mathcal{F}_{\mathbb{P}}(n)$ into $[0,1]$, that is to say, product logic valuations. Thus, the state space results to be generated by the truth-value assignments of the logic.

Finally, Section 6 is devoted to presenting a logic for probabilistic reasoning on many-valued events represented by formulas of product logic. For that formalism we will provide an (infinitary) axiomatization which is sound and complete with respect to a probabilistic-like semantics given by states of free product algebras.

For the sake of readability we moved some technical proofs in an appendix at the end of the paper.

2. Product algebras and product functions

In this section we are going to recall some basic facts and preliminary notions about product algebras. In particular we will focus on free, finitely generated, product algebras and their functional representation, mainly reporting results from [3]. We assume the reader to be familiar with standard notions of universal algebra and algebraic semantics for many-valued logics. For otherwise we point them to the standard monographs [8] and [21,12] respectively. To start with, let us recall that a BL-algebra [21] is a bounded, integral and commutative residuated lattice $\mathbf{A} = (A, \odot, \to, \wedge, \vee, 0, 1)$ which satisfies the following equations:

[1] Note that, unlike Kroupa–Panti theorem, we do not deal with states of *any* product algebra but of finitely-generated free product algebras.