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A general unified framework for interval pairwise comparison matrices

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ABSTRACT

Interval Pairwise Comparison Matrices have been widely used to account for uncertain statements concerning the preferences of decision makers. Several approaches have been proposed in the literature, such as multiplicative and fuzzy interval matrices. In this paper, we propose a general unified approach to Interval Pairwise Comparison Matrices, based on Abelian linearly ordered groups. In this framework, we generalize some consistency conditions provided for multiplicative and/or fuzzy interval pairwise comparison matrices and provide inclusion relations between them. Then, we provide a concept of distance between intervals that, together with a notion of mean defined over real continuous Abelian linearly ordered groups, allows us to provide a consistency index and an indeterminacy index. In this way, by means of suitable isomorphisms between Abelian linearly ordered groups, we will be able to compare the inconsistency and the indeterminacy of different kinds of Interval Pairwise Comparison Matrices, e.g. multiplicative, additive, and fuzzy, on a unique Cartesian coordinate system.

1. Introduction

As their name suggests, Pairwise Comparison Matrices (PCMs) have been a long standing technique for comparing alternatives and their role has been pivotal in the development of modern decision making methods. In accordance with decision theory, in this paper we shall consider a finite non-empty set of n entities (e.g. criteria or alternatives) $X = \{x_1, \ldots, x_n\}$, and the object of our investigation is the set of comparisons between them with respect to one of their properties. That is, we are interested in the subjective estimations a_{ij} $\forall i, j \in \{1, \ldots, n\}$, where a_{ij} is a numerical representation of the intensity of preference of x_i over x_j.

With respect to the values that a_{ij} can assume and their interpretation, it is fundamental to be aware that various proposals have been presented, studied, and applied in the literature to solve real-world problems.

The foremost type of representation of valued preferences, at least with respect to the number of real-world applications is probably the multiplicative representation, used among others by Saaty in the theory of the Analytic Hierarchy Process (AHP). In this sense, pairwise comparisons are expressed as positive real numbers, $a_{ij} \in]0, +\infty[$ satisfying a condition of multiplicative reciprocity, $a_{ij} \cdot a_{ji} = 1$. We shall note that the AHP [37] is not the only method using this scheme for pairwise

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comparisons. For instance, proponents of Multi Attribute Value Theory (MAVT) such as Keeney and Raiffa [24] and Belton and Stewart [2] advocate the use of pairwise comparisons to estimate the ratios between weights of criteria when the value function is additive. Hereafter, representations of preference of this kind will be called *multiplicative*.

Reciprocal preference relations, whose origins can be traced back at least to a study by Zermelo [53], assume that intensities of preferences are represented on the open unit interval; that is, $a_{ij} \in]0,1[$. Similarly to the multiplicative case, reciprocal preference relations obey a condition of reciprocity, in this case $a_{ij} + a_{ji} = 1$. Interestingly, such a representation was studied, among others, also by Luce and Suppes [31] under the name of 'probabilistic preference relations' and has been widely popularized within the fuzzy sets community under the name of 'fuzzy preference relations'. Instances of influential studies on these mathematical structures under the fuzzy lens have been offered by Tanino [40], Herrera-Viedma et al. [21] and Kacprzyk [23]. For sake of simplicity, and because the unit interval recalls the idea of membership function, we shall refer to this case as the *fuzzy* case in the rest of this manuscript.

The third representation considered in this paper shall be called *additive* due to the fact that intensities of preferences are expressed as real numbers, $a_{ij} \in]-\infty, +\infty[$ and comply with a condition of additive reciprocity, i.e. $a_{ij} + a_{ji} = 0$. We shall note that this representation coincides with the Skew-symmetric additive representation of utilities proposed by Fishburn [17] and with the representation used by some decision analysis methodologies such as REMBRANDT [35].

All in all, it emerges a picture where the technique of pairwise comparisons plays an important role within decision theory. Moreover, in spite of their different formulations and interpretations, it was formalized that different representations share the same algebraic structure [8], based on Abelian linearly ordered groups, i.e. commutative groups equipped with an ordering relation. Hence, to derive results which are general enough to pertain to each of these representations of preferences, we will focus on this more general algebraic representation and exploit the full potential of group theory. Several authors have already adopted this approach based on Abelian linearly ordered groups (e.g. [22,25,36,47]). Nevertheless, in spite of the general formulation of our results, examples involving specific representations of preferences will be used in the rest of this paper.

More specifically, within this framework, we shall investigate the case of Interval Pairwise Comparisons Matrices (IPCMs) according to which comparison values are expressed as intervals $\tilde{a}_{ij} = [a_{ij}^-, a_{ij}^+] \subset \mathbb{R}$ instead of real numbers. The approach with intervals has been widely used to account for uncertain statements concerning the preferences of a decision maker (e.g. [26,54]) and studied separately in the case of multiplicative preference relations [38,39] and in the case of fuzzy preference relations [49], just to cite few examples. In this paper, we shall generalize it and derive broader results. More specifically, we will generalize interval arithmetic and propose a concept of metric on intervals when these are subsets of Abelian linearly ordered groups. This will be instrumental to formulate the concept of IPCM and study, in a more general context, the notions of reciprocity, consistency, and indeterminacy. Having done this, we will propose and justify a consistency index which, in concert with an indeterminacy index, can be used to evaluate the acceptability of IPCMs.

There are further papers in the literature that take into consideration consistency and indeterminacy: Wang [43] considered multiplicative IPCMs and proposed a geometric mean based uncertainty index to capture the inconsistency in the original multiplicative PCM; Liu [28] measured the consistency of a multiplicative IPCM by computing Saaty's consistency index [37] of one or two associated PCMs; Li et al. [27] and Wang and Chen [44] proposed as indeterminacy index the geometric mean of the ratios $\frac{a_{ij}^+}{a_{ij}^-}$ for both multiplicative and fuzzy IPCMs. However, no paper proposed a consistency index to be computed directly from the IPCM, i.e. without considering associated PCMs, and no paper proposes both a general consistency index and a general indeterminacy index suitable for each kind of IPCM (e.g. additive, multiplicative, and fuzzy).

The paper is organized as follows. Section 2 provides the necessary notions and notation for the real-valued case. Next, in Section 3, we discuss the idea of intervals defined over a special type of group structure. By drawing from the previous two sections, in Section 4 we present a general notion of interval pairwise comparison matrix which has the merit of unifying different approaches under the same umbrella. This will give us the possibility, in Section 5, to discuss reciprocity and consistency conditions in a more general setting. In Sections 6 and 7, we introduce a consistency and an indeterminacy index, respectively. These indices can be used in concert to evaluate the acceptability of preferences. Section 8 draws some conclusions and proposes directions for future work. Finally, Appendix contains the proofs of the statements

2. Notation and preliminaries

In this section, we will provide notation and preliminaries which will be necessary in the rest of the paper.

2.1. Abelian linearly ordered groups

We start providing definitions and essential notation about Abelian linearly ordered groups in order to define Pairwise Comparison Matrices (Subsection 2.2) and Interval Pairwise Comparison Matrices (Sections 4 and 5); for further details the reader can refer to [8].

Definition 2.1. Let G be a non-empty set, $\odot : G \times G \to G$ a binary operation on G, $\leq$ a weak order on G. Then, $\mathcal{G} = (G, \odot, \leq)$ is an *Abelian linearly ordered group*, *Alo-group* for short, if $(G, \odot)$ is an Abelian group and

$$a \leq b \Rightarrow a \odot c \leq b \odot c. \tag{1}$$