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Idempotent conjunctive and disjunctive combination of belief functions by distance minimization ☆

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A B S T R A C T

Idempotence is a desirable property when cautiousness is wanted in an information fusion process, since in this case combining identical information should not lead to the reinforcement of some hypothesis. Idempotent operators also guarantee that identical information items are not counted twice in the fusion process, a very important property in decentralized applications where the information origin cannot always be tracked (ad-hoc wireless networks are typical examples). In the theory of belief functions, a sound way to combine conjunctively multiple information items is to design a combination rule that selects the least informative element among a subset of belief functions more informative than each of the combined ones. In contrast, disjunctive rules can be retrieved by selecting the most informative element among a subset of belief functions less informative than each of the combined ones. One interest of such approaches is that they provide idempotent rules by construction.

The notions of less and more informative are often formalized through partial orderings extending usual set-inclusion, yet the only two informative partial orders that provide a straightforward idempotent rule leading to a unique result are those based on the conjunctive and disjunctive weight functions. In this article, we show that other partial orders can achieve a similar goal when the problem is slightly relaxed into a distance optimization one. Building upon previous work, this paper investigates the use of distances compatible with informative partial orders to determine a unique solution to the combination problem. The obtained operators are conjunctive/disjunctive, idempotent and commutative, but lack associativity. They are, however, quasi-associative allowing sequential combinations at no extra complexity. Some experiments demonstrate interesting discrepancies as compared to existing approaches, notably with the aforementioned rules relying on weight functions.

1. Introduction

The theory of belief functions is a framework for reasoning under uncertainty. It was initially proposed to model imprecise statistical observation [1], and this initial work was then extended [2] to include subjective or epistemic uncertainty

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(*e.g.*, when a variable has a fixed, yet ill-known value). It has received a considerable attention in the soft computing community as it allows the combination of uncertain, imprecise or conflictual pieces of evidence. The flexibility of the theory of belief functions has led some people to think of it as a data fusion framework whereas its initial purpose is more general.

Combining pieces of evidence coming from different sources of information is one of the most frequently studied problem in the belief function theory. In particular, a rich literature exists (see for example [3,4] and references therein) proposing alternatives to Dempster's rule when this latter does not apply, that is when sources of information are either unreliable or non-independent, or both. This paper deals with the second issue, that is the one concerning source independence, and more particularly with the case where this dependence is ill-known and hard to assess.

Under such an assumption, it is common to adopt a cautious approach, also known as least-commitment principle [5] (LCP). A natural consequence of this principle is that if all the sources provide the same mass function, then the result of the combination should be this very mass function, or in other words the combination should be idempotent. However, if idempotence is a consequence of the LCP, satisfying idempotence does not imply satisfying the LCP. As shown by Dubois and Yager [6], there is virtually an infinity of ways to derive idempotent combination rules, not all of them necessarily following a least-commitment principle. For instance, Cattaneo [7] provides an idempotent rule following a conflict-minimization approach, which may lead to non-least committed results [8].

So, to satisfy the LCP, we must add additional constraints on the combination rule. One such natural constraint is to consider a partial order over informative content of mass functions, and to require the combination result to be one of the maximal elements of this partial order within the subset of possible combination results. Unfortunately, such an approach can present two shortcomings: it will very often lead to multiple solutions corresponding to all possible maximal elements [9], and estimating this set of solutions may be computationally challenging. Denœux [10] shows that using the canonical decomposition and the associated partial order leads to a unique LCP, idempotent solution, yet this solution has two limitations: the set of possible combination results is quite small, leading to a not so conservative behavior (as we will see on a simple example in Section 5, and as already pointed out in [8]), and the combination only apply to specific (*i.e.*, non-dogmatic) mass functions.

In this paper, we take inspiration from some of our previous work [11] studying the consistency of distances with partial orders comparing informative contents to propose a new way to derive cautious combination rules. Our approach departs from previous ones, as it is formulated as an optimization problem (similarly to what is done by Cattaneo [7] for conflict minimization) that naturally satisfies the LCP principle. Our approach makes minimal assumptions about the shape of the combination result, in the sense that the only constraints it imposes on the combination result is to be more informative than each initial belief function in the conjunctive case, and less informative in the disjunctive case. This also contrasts with previous approaches [10,12,8], that considered the results to take specific forms (either in the form of a joint mass function with prescribed marginals [12,8], or in a weight function combined through uninorms [10]). It is in fact in-line with the generic conjunctive operator described by Dubois et al. [4].

Our approach also solves the two problems of solution uniqueness and computability, since if the distance is chosen so as to minimize a strictly convex objective function, we are guaranteed to have a unique solution satisfying the LCP and computable by convex optimization. Section 2 recalls the basics needed in this paper. The bulk of the proposal is contained in Section 3, where we present the combination approach and study its properties in the conjunctive case. In section 4, we present equivalent results for the disjunctive case. Section 5 compares our proposal with respect to existing ones.

This paper is an extended version of [13] which was presented at the 4th international conference on belief functions, BELIEF'16 to which this special issue of IJAR is dedicated.

2. Preliminaries and problem statement

This section briefly sketches the basics of evidence theory and provides references for readers interested in further details. Like most of the belief function literature, this paper is limited to belief functions on finite spaces. The derivation of the results introduced in this paper in the continuous case is left for future work.

2.1. Basic concepts

A body of evidence $\mathcal{E}_i$ defined on the finite space $\Omega = \{\omega_1, \dots, \omega_n\}$ will be modeled by a mass function $m_i : 2^\Omega \to [0, 1]$ that sums up to one, *i.e.*, $\sum_{E \subseteq \Omega} m_i(E) = 1$. Following usual notation, 2^Ω denotes the power set of Ω. In evidence theory, this basic tool models our uncertainty about the true value of some quantity (parameter, variable) lying in Ω. The cardinality of 2^Ω is denoted by $N = 2^n$. The set $\mathcal{M}$ of mass functions on Ω is called **mass space**. A set A is a **focal element** of m iff $m(A) > 0$. The **complement** $\overline{m}$ of a mass function m is such that $\forall A \subseteq \Omega$, we have $\overline{m}(A) = m\left(A^c\right)$ where A^c denotes the complement of the set A in Ω.

A mass function assigning a unit mass to a single focal element A is called **categorical** and denoted by m_A: $m_A(A) = 1$. If $A \neq \Omega$, the mass function m_A is equivalent to providing the set A as information, while the **vacuous** mass function m_Ω represents ignorance. A function m_i such that $m_i = (1 - \alpha)m_A + \alpha m_\Omega$ with $\alpha \in [0; 1]$ is called a **simple** mass function and is regarded as an elementary evidence supporting the event A.

Besides, a mass function m_i such that $m_i(\Omega) = 0$, *i.e.* Ω is not a focal element of m_i, is called a **dogmatic** mass function. A mass function m_i such that $m_i(\emptyset) = 0$, *i.e.* $\emptyset$ is not a focal element of m_i, is called a **normalized** mass function while