International Journal of Approximate Reasoning ••• (••••) •••–•••

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

A new definition of order relation for the introduction of algebraic fuzzy closure operators

Bin Pang [a,b], Yi Zhao [b,∗], Zhen-Yu Xiu [c]

[a] School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China
[b] Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen 518055, PR China
[c] College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610000, PR China

ARTICLE INFO

Article history:
Received 1 November 2016
Received in revised form 9 June 2017
Accepted 4 July 2017
Available online xxxx

Keywords:
Fuzzy set
Fuzzy closure system
Fuzzy closure operator
Fuzzy convex structure

ABSTRACT

In this paper, a new approach to order relation between fuzzy sets is provided, which is called well inclusion order between fuzzy sets. Based on this new order relation, the concept of algebraic fuzzy closure operators is introduced. It is shown that there is a categorical isomorphism between algebraic fuzzy closure operators and fuzzy convex structures. Also, the relationship between fuzzy closure systems and fuzzy convex structures is investigated. It is proved that the category of fuzzy convex spaces is a bicoreflective subcategory of the category of fuzzy closure system spaces.

1. Introduction

Since Zadeh introduced fuzzy sets [35] in control theory, fuzzy set theory has deserved wide attention. Up to now, fuzzy set theory has affected many areas of scientific research (sometimes marginally, sometimes significantly) ranging from mathematics (especially many-valued logics, topology, algebra and category theory) to engineering practice especially in modeling, control, optimization, and data processing, also with some clear impact on techniques devoted to pattern recognition and image processing, operations research, artificial intelligence, databases and information systems. In each research area, the problem of ordering fuzzy sets should be addressed. As a natural generalization of inclusion relation between classical sets, the inclusion order between fuzzy sets can be defined in a natural way. However, in some aspect, the inclusion order between fuzzy sets is not the most suitable counterpart of inclusion order between classical sets. In this situation, new forms of counterparts of inclusion order between classical sets have to be proposed. In [18], John et al. defined fuzzy subsethood for fuzzy sets and used Zadeh's extension principle to extend Kosko's definition of the fuzzy subsethood measure to type-2 fuzzy sets [23]. Later, Bustince et al. [9] provided a new approach to order interval-valued fuzzy sets and combined it with fuzzy Choquet intervals ([17,19]). In a different way, Bělohlávek ([4,5]) introduced fuzzy inclusion order between L-subsets, where L is a complete residuated lattice, and used it to define L_K-closure operators and L_K-closure systems. Adopting the definition of fuzzy inclusion order, Fang ([11,12]) and Li [21] independently proposed the concept of stratified L-ordered convergence structures.

With the development of fuzzy set theory, various particular fuzzy closure operators appear in several areas of fuzzy logic and its applications, including fuzzy relational equations, fuzzy mathematical morphology, approximate reasoning, and fuzzy logic in narrow sense and its applications such as fuzzy logic programming or formal concept analysis of data with

∗ Corresponding author.
E-mail addresses: pangbin1205@163.com (B. Pang), zhaoyisz420@sohu.com (Y. Zhao), xiuzhenyu112@sohu.com (Z.-Y. Xiu).

http://dx.doi.org/10.1016/j.ijar.2017.07.003
0888-613X/© 2017 Published by Elsevier Inc.

fuzzy attributes (see [3,4], [7,8], [10], [13,14], [16] and [24]). This paper is related to fuzzy closure operators in a profound way, namely, fuzzy closure operators and related structures, such as fuzzy convex structures (see [22,25,26,28]). Recall that an algebraic closure operators (also called domain finite closure operator in [31]) can be used to characterize convex structures due to its domain-finiteness. Considering the characterizations of fuzzy convex structures, algebraic fuzzy closure operators, as a counterpart of algebraic closure operators, should be emphasized. Since algebraic fuzzy closure operators cannot be defined by the inclusion order or the fuzzy inclusion order, the problem of defining algebraic fuzzy closure operators reduces to the problem of introducing a new order relation between fuzzy sets.

The contributions of this paper are the following. Firstly, we define a new order relation between fuzzy sets and called it well inclusion order between fuzzy sets. Then we investigate its basic properties in preparations for the following two sections. Secondly, by means of well inclusion order, we define algebraic fuzzy closure operators and establish its categorical relationship with fuzzy convex structures. Finally, we discuss the relationship between fuzzy convex structures and fuzzy closure systems. We can show the category of fuzzy convex spaces is a bicoreflective subcategory of the category of fuzzy closure system spaces.

2. Preliminaries

This section provides basic notions from fuzzy sets, fuzzy closure operators, category theory and fuzzy convex structures. More details can be found, e.g., in [14], [16] and [20] (fuzzy sets), [4,6], [14,15] and [34] (fuzzy closure operators), [2] and [27] (category theory), [22,28–30,32] and [33] (fuzzy convex structures).

2.1. Fuzzy sets

Throughout this paper, let X be a nonempty set and let 2^X be the powerset of X. Denote the set of all mappings from X to $[0,1]$ by $\mathcal{F}(X)$. Each element of $\mathcal{F}(X)$ is called a fuzzy set. For $A, B \in \mathcal{F}(X)$, we call A less than or equal to B provided that $A(x) \leqslant B(x)$ for each $x \in X$, denoted by $A \leqslant B$. In this way, we call $\leqslant$ the inclusion order between fuzzy sets. With the inclusion order, $(\mathcal{F}(X), \leqslant)$ is a complete lattice. For $A, B \in \mathcal{F}(X)$ and $\mathcal{A} \subseteq \mathcal{F}(X)$, we denote by $A \wedge B$, $A \vee B$, $\bigvee \mathcal{A}$ and $\bigwedge \mathcal{A}$ the fuzzy sets defined by $(A \wedge B)(x) = \min\{A(x), B(x)\}$, $(A \vee B)(x) = \max\{A(x), B(x)\}$, $(\bigvee \mathcal{A})(x) = \sup_{A \in \mathcal{A}} A(x)$ and $(\bigwedge \mathcal{A})(x) = \inf_{A \in \mathcal{A}} A(x)$. The smallest element and the largest element in $\mathcal{F}(X)$ are denoted by 0_X and 1_X, respectively.

For each $A \in \mathcal{F}(X)$, A is called finite provided that the support set of A is finite. That is, $Supp(A) = \{x \in X \mid A(x) \neq 0\}$ is finite. Let $\mathcal{F}_{fin}(X)$ denote the set of all finite fuzzy sets of X.

We say $\{A_j\}_{j \in J}$ is a directed subset of $\mathcal{F}(X)$, in symbols $\{A_j\}_{j \in J} \overset{dir}{\subseteq} \mathcal{F}(X)$, if for each $A_{j_1}, A_{j_2} \in \{A_j\}_{j \in J}$, there exists $A_{j_3} \in \{A_j\}_{j \in J}$ such that $A_{j_1}, A_{j_2} \leqslant A_{j_3}$.

Let X, Y be two nonempty sets and $\varphi : X \longrightarrow Y$ be a mapping. Define $\varphi^{\rightarrow} : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ and $\varphi^{\leftarrow} : \mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$ by $\varphi^{\rightarrow}(A)(y) = \sup_{\varphi(x)=y} A(x)$ for $A \in \mathcal{F}(X)$ and $y \in Y$, and $\varphi^{\leftarrow}(B) = B \circ \varphi$ for $B \in \mathcal{F}(Y)$, respectively.

2.2. Fuzzy closure operators and category theory

For a nonempty set X, a (Birkhoff) closure operator on X is a mapping $c : 2^X \longrightarrow 2^X$ which satisfies:

(CL1) $c(\emptyset) = \emptyset$;
(CL2) $U \subseteq c(U)$;
(CL3) $c(c(U)) = c(U)$;
(CL4) $U \subseteq V$ implies $c(U) \subseteq c(V)$.

An algebraic (also called domain finite in [31]) closure operator on X is a closure operator c on X which satisfies:

(ACL) $c(U) = \bigcup\{c(F) \mid F \subseteq U \text{ and } F \text{ is finite}\}$.

For an algebraic closure operator c on X, the pair (X, c) is called an algebraic closure space.

Considering the fuzzy counterpart of closure operators, (CL1)–(CL4) can be directly generalized to the fuzzy case.

Definition 2.1. A fuzzy closure operator on X is a mapping $\mathfrak{c} : \mathcal{F}(X) \longrightarrow \mathcal{F}(X)$ which satisfies:

(FCL1) $\mathfrak{c}(0_X) = 0_X$;
(FCL2) $A \leqslant \mathfrak{c}(A)$;
(FCL3) $\mathfrak{c}(\mathfrak{c}(A)) = \mathfrak{c}(A)$;
(FCL4) $A \leqslant B$ implies $\mathfrak{c}(A) \leqslant \mathfrak{c}(B)$.

For a fuzzy closure operator $\mathfrak{c}$ on X, the pair $(X, \mathfrak{c})$ is called a fuzzy closure space.

Please cite this article in press as: B. Pang et al., A new definition of order relation for the introduction of algebraic fuzzy closure operators, Int. J. Approx. Reason. (2017), http://dx.doi.org/10.1016/j.ijar.2017.07.003