International Journal of Approximate Reasoning 92 (2018) 97–119

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Lexicographic choice functions

Arthur Van Camp [a,*], Gert de Cooman [a], Enrique Miranda [b]

[a] Ghent University, IDLab, Technologiepark-Zwijnaarde 914, 9052 Zwijnaarde, Belgium
[b] Department of Statistics and Operations Research, University of Oviedo, Spain

ARTICLE INFO

Article history:
Received 23 May 2017
Received in revised form 16 August 2017
Accepted 7 October 2017
Available online 6 November 2017

Keywords:
Choice functions
Coherence
Lexicographic probabilities
Horse lotteries
Maximality
Sets of desirable gambles

ABSTRACT

We investigate a generalisation of the coherent choice functions considered by Seidenfeld et al. [35], by sticking to the convexity axiom but imposing no Archimedeanity condition. We define our choice functions on vector spaces of options, which allows us to incorporate as special cases both Seidenfeld et al.'s [35] choice functions on horse lotteries and also pairwise choice—which is equivalent to sets of desirable gambles [29]—, and to investigate their connections.

We show that choice functions based on sets of desirable options (gambles) satisfy Seidenfeld's convexity axiom only for very particular types of sets of desirable options, which are exactly those that are representable by lexicographic probability systems that have no non-trivial Savage-null events. We call them lexicographic choice functions. Finally, we prove that these choice functions can be used to determine the most conservative convex choice function associated with a given binary relation.

1. Introduction

Since the publication of the seminal work of Arrow [3] and Uzawa [40], coherent choice functions have been used widely as a model of the rational behaviour of an individual or a group. In particular, Seidenfeld et al. [35] established an axiomatisation of coherent choice functions, generalising Rubin's [31] axioms to allow for incomparability. Under this axiomatisation, they proved a representation theorem for coherent choice functions in terms of probability–utility pairs: a choice function C satisfies their coherence axioms if and only if there is some non-empty set S of probability–utility pairs such that $f \in C(A)$ whenever the option f maximises p-expected u-utility over the set of options A for some (p, u) in S.

Allowing for incomparability between options may often be of crucial importance. Faced with a choice between two options, a subject may not have enough information to establish a (strict or weak) preference of one over the other: the two options may be incomparable. This will indeed typically be the case when the available information is too vague or limited. It arises quite intuitively for group decisions, but also for decisions made by a single subject, as was discussed quite thoroughly by Williams [45], Levi [24], and Walley [43], amongst many others. Allowing for incomparability lies at the basis of a generalising approach to probability theory that is often referred to by the term *imprecise probabilities*. It unifies a diversity of well-known uncertainty models, including typically non-linear (or non-additive) functionals, credal sets, and sets of desirable gambles; see the introductory book by Augustin et al. [4] for a recent overview. Among these, coherent sets of desirable gambles, as discussed by Quaeghebeur [29], are usually considered to constitute the most general and powerful type of model. Such sets collect the gambles that a given subject considers strictly preferable to the status quo.

* Corresponding author.
E-mail addresses: arthur.vancamp@ugent.be (A. Van Camp), gert.decooman@ugent.be (G. de Cooman), mirandaenrique@uniovi.es (E. Miranda).

https://doi.org/10.1016/j.ijar.2017.10.011
0888-613X/

Nevertheless, choice functions clearly lead to a still more general model than sets of desirable gambles, because the former's preferences are not necessarily completely determined by the pair-wise comparisons between options that essentially constitute the latter. This was of course already implicit in Seidenfeld et al.'s [35] work, but was investigated in detail in one of our recent papers [42], where we zoomed in on the connections between choice functions, sets of desirable gambles, and indifference.

In order to explore the connection between indifference and the strict preference expressed by choice functions, we extended the above-mentioned axiomatisation by Seidenfeld et al. [35] to choice functions defined on vector spaces of options, rather than convex sets of horse lotteries, and also let go of two of their axioms: (i) the Archimedean one, because it prevents choice functions from modelling the typically non-Archimedean preferences captured by coherent sets of desirable gambles; and (ii) the convexity axiom, because it turns out to be hard to reconcile with Walley–Sen maximality as a decision rule, something that is closely tied in with coherent sets of desirable options [38]. Although our alternative axiomatisation allows for more leeway, and for an easy comparison with the existing theory of sets of desirable gambles, it also has the drawback of no longer forcing a representation theorem, or in other words, of not leading to a strong belief structure (we refer to De Cooman [13] for a more detailed discussion of belief models that constitute a strong belief structure). Such a representation is nevertheless interesting because strong belief structures have the advantage that their coherent models are infima (under a partial order implicit in the structure) of their dominating *maximally informative* models. This allows for reasoning with the (typically simpler) maximally informative dominating models, instead of the (possibly more complex) models themselves. In an earlier paper [42], we discussed a few interesting examples of special 'representable' choice functions, such as the ones from a coherent set of desirable gambles via maximality, or those determined by a set of probability measures via E-admissibility.

The goal of the present paper is twofold: to (i) further explore the connection of our definition of choice functions with Seidenfeld et al.'s [35]; and to (ii) investigate in detail the implications of Seidenfeld et al.'s [35] convexity axiom in our context. We will prove that, perhaps somewhat surprisingly, for those choice functions that are uniquely determined by binary comparisons, convexity is equivalent to being representable by means of a lexicographic probability measure. This is done by first establishing the implications of convexity in terms of the binary comparisons associated with a choice function, giving rise to what we will call *lexicographic sets of desirable gambles*. These sets include as particular cases the so-called *maximal* and *strictly desirable* sets of desirable gambles. Although in the particular case of binary possibility spaces these are the only two possibilities, for more general spaces lexicographic sets of gambles allow for a greater level of generality, as one would expect considering the above-mentioned equivalence.

A consequence of our equivalence result is that we can consider infima of choice functions associated with lexicographic probability measures, and in this manner subsume the examples of E-admissibility and M-admissibility discussed by Van Camp et al. [42]. It will follow from the discussion that these infima also satisfy the convexity axiom. As one particularly relevant application of these ideas, we prove that the most conservative convex choice function associated with a binary preference relation can be obtained as the infimum of its dominating lexicographic choice functions.

The paper is organised as follows. In Section 2, we recall the basics of coherent choice functions on vector spaces of options as introduced in our earlier work [41]. We motivate our definitions by showing in Section 3 that they include in particular coherent choice functions on horse lotteries, considered by Seidenfeld et al.'s [35], and we discuss in some detail the connection between the rationality axioms considered by Seidenfeld et al. [35] and ours.

As a particularly useful example, we discuss in Section 4 those choice functions that are determined by binary comparisons. We have already shown before [42] that this leads to the model of coherent sets of desirable gambles; here we study the implications of including convexity as a rationality axiom.

In Section 5, we motivate our definition of lexicographic choice functions and study the properties of their associated binary preferences. We prove the connection with lexicographic probability systems and show that the infima of such choice functions can be used when we want to determine the implications of imposing convexity and maximality. We conclude with some additional discussion in Section 6.

2. Coherent choice functions on vector spaces

Consider a real vector space $\mathcal{V}$ provided with the vector addition $+$ and scalar multiplication. We denote the additive identity by 0. For any subsets A_1 and A_2 of $\mathcal{V}$ and any λ in $\mathbb{R}$, we let $\lambda A_1 := \{\lambda u : u \in A_1\}$ and $A_1 + A_2 := \{u + v : u \in A_1$ and $v \in A_2\}$.

Elements of $\mathcal{V}$ are intended as abstract representations of *options* amongst which a subject can express his preferences, by specifying choice functions. Often, options will be real-valued maps on some possibility space, interpreted as uncertain rewards—and therefore also called *gambles*. More generally, they can be *vector-valued gambles*: vector-valued maps on the possibility space. We will see further on that by using such vector-valued gambles, we are able to include as a special case *horse lotteries*, the options considered for instance by Seidenfeld et al. [35]. Also, we have shown [42] that indifference for choice functions can be studied efficiently by also allowing equivalence classes of indifferent gambles as options; these yet again constitute a vector space, where now the vectors cannot always be identified easily with maps on some possibility space, or gambles. For these reasons, we allow in general any real vector space to serve as our set of (abstract) possible options. We will call such a real vector space an *option space*.