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Bivariate quadratic copula constructions ☆

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ABSTRACT

In this study, we present the concept of quadratic copula constructions using two arbitrary copulas. We characterize all quadratic polynomials of four variables whose composition with any two copulas always results in a copula. We show that these polynomials form a compact convex set in a seven-dimensional vector space. We also apply the result to obtain a new family of copulas.

1. Introduction and preliminaries

A (bivariate) copula can be viewed as a joint distribution of two random variables uniformly distributed on the unit interval $\mathbb{I} = [0, 1]$. Equivalently, a *copula* is a function $C : \mathbb{I}^2 \to \mathbb{I}$ satisfied the following conditions.

(G) $C(0, u) = 0 = C(u, 0)$ for all $u \in \mathbb{I}$.
(U) $C(1, u) = u = C(u, 1)$ for all $u \in \mathbb{I}$.
(2I) $V_C([u_1, u_2] \times [v_1, v_2]) \geq 0$ for all $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$ where

$$V_C([u_1, u_2] \times [v_1, v_2]) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \tag{1.1}$$

is the C-volume of the rectangle $[u_1, u_2] \times [v_1, v_2]$.

Denote the set of all copulas by $\mathcal{C}$. Examples of well-known copulas are

$$W(u, v) = \max(0, u + v - 1),$$
$$\Pi(u, v) = uv, \qquad \text{and}$$
$$M(u, v) = \min(u, v)$$

which are called the *Frechet–Hoeffding lower bound*, the *product copula*, and the *Frechet–Hoeffding upper bound*, respectively. Note that for every copula C,

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$$W(u,v) \le C(u,v) \le M(u,v) \tag{1.2}$$

for all $u, v \in \mathbb{I}$. Moreover, a function $C : \mathbb{I}^2 \to \mathbb{I}$ is a copula if and only if it satisfies equation (1.2) and **(2I)**.

Another class of copulas crucial for this work is the shuffle of min, a copula C is an $m \times m$ *(straight) shuffle of min by a bijection j on* $\{0, 1, \ldots, m-1\}$ if $V_C\left(\left[\frac{i}{m}, u\right] \times \left[\frac{j(i)}{m}, v\right]\right) = \min\left(u - \frac{i}{m}, v - \frac{j(i)}{m}\right)$ for any $u \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$, $v \in \left[\frac{j(i)}{m}, \frac{j(i)+1}{m}\right]$ and $i = 0, \ldots, m-1$. This implies $V_C\left(\left[\frac{i}{m}, \frac{i+1}{m}\right] \times \left[\frac{j}{m}, \frac{j+1}{m}\right]\right) = 0$ whenever $j \ne j(i)$ and also an $m \times m$ shuffle of min is a $k \times k$ shuffle of min whenever m divides k. Note that the set of shuffles of min is dense in the set of copulas $\mathcal{C}$ under uniform convergence [1].

Since the concept of copulas was introduced in 1959, many copula families have been introduced including ones obtained by transformations of copulas. The simplest example is probably the copula transpose defined by

$$\tau_t(C)(u,v) = C(v,u)$$

for all $u, v \in \mathbb{I}$. Then $\tau_t(C)$ is a copula whenever C is, that is, τ_t is a function on $\mathcal{C}$. Other simple transformations based on the symmetry of uniform distribution are τ_u and τ_v defined by

$$\tau_u(C)(u,v) = u - C(u, 1-v), \text{ and}$$
$$\tau_v(C)(u,v) = v - C(1-u, v)$$

for all $u, v \in \mathbb{I}$ and $C \in \mathcal{C}$. Multivariate transformations of copulas also exists. For example, the convex combinations $\kappa_t(C, D) = tC + (1-t)D$ of copulas C and D are copulas for all $t \in \mathbb{I}$. Others include ordinal sums and patchwork copulas [2–9].

In 2015, Kolesárová et al. [10] characterizes all transformations of the form $\xi_P(C)(u,v) = P(u, v, C(u,v))$ where P is a quadratic polynomial. They show that ξ_P is a copula transformation if and only if P can be written as

$$P(u,v,w) = auv + (1+b-a)w - buw - bvw + bw^2 \tag{1.3}$$

where both $a \in \mathbb{I}$ and $a - b \in \mathbb{I}$. It is natural to ask whether multivariate versions of this type of transformations exist and if they do, whether we can characterize them. In this work, we provide an answer for the bivariate version, that is, we characterizes all quadratic polynomial $P = P(u, v, w_1, w_2)$ such that ζ_P defined by $\zeta_P(C, D)(u,v) = P(u, v, C(u,v), D(u,v))$ is a function from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$. It turns out that characterizing them is quite challenging since the class of ζ_P actually form a seven-dimensional convex set with its boundary consists of 24 hyperplane sections. We have to rely on symmetry method to reduce the number of variables. However, we are unable to classify all of its extreme points. It is also interesting whether this approach can be extended to the multivariate version.

In the next section, we provide the statement of the main result of this work follows by some examples. The proof of this main result will be, however, postponed until section 3.

2. Main results

Since the linear transformation $(w_1, w_2) \mapsto \left(\frac{1}{2}(w_1 + w_2), \frac{1}{2}(w_1 - w_2)\right)$ is invertible, any quadratic polynomial can be written as a quadratic function of $u, v, \frac{1}{2}(w_1 + w_2), \frac{1}{2}(w_1 - w_2)$ instead of u, v, w_1, w_2. Thus, we will characterize all quadratic polynomials P such that ζ_P defined by

$$\zeta_P(C,D)(u,v) = P\left(u, v, \frac{1}{2}\left(C(u,v) + D(u,v)\right), \frac{1}{2}\left(C(u,v) - D(u,v)\right)\right) \tag{2.1}$$

is a bivariate copula transformation. It is clear that this characterization is equivalent to the original formulation of the problem. In fact, we will show in section 3 that such $P = P(u, v, z_1, z_2)$ must be in the form

$$\begin{aligned} P(u,v,z_1,z_2) &= (1-a)uv + bz_1 + (a-b)z_1(u+v-z_1) \\ &\quad + cz_2 + dz_2^2 + ez_1z_2 + (fu + gv)z_2 \end{aligned} \tag{2.2}$$

where $a, b \in \mathbb{I}$ and $c, d, e, f, g \in \mathbb{R}$ are constants satisfying the following conditions.

(A1) $-2 + a + b \le f \le 2 - a - b$ and $a + b - 2 \le g \le 2 - a - b$
(A2) $-b \le c \le b$
(A3) $-a - b - d \le 2c + e + f + g \le a + b + d$
(A4) $-2 + a + b + d \le f - g \le 2 - a - b - d$
(A5) $-2 + a + b \le e + f \le 2 - a - b$ and $a + b - 2 \le e + g \le 2 - a - b$
(A6) $-2 + a + b - d \le e + f + g \le 2 - a - b + d$
(A7) $-a \le c + f \le a$ and $-a \le c + g \le a$
(A8) $-b \le c + e + f + g \le b$
(A9) $-a - b + d \le 2c + f + g \le a + b - d$