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Covering-based rough sets and modal logics. Part I

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ABSTRACT

Two conjectures on the covering-based rough set semantics for modal logics in [35] are answered. The C_2 and C_5 semantics give rise to the same modal system **S4**. There are Galois connections between C_2 and C_5 which lead to the covering-based semantics for the temporal logic system **S4***t*. The P_1 and C_4 semantics give rise to the same modal system **KTB**.

1. Introduction

Rough set theory, proposed in 1982 by Pawlak [29], is apparently related to the modal logic system **S5**. A Pawlakian approximation space (X, R), where X is a non-empty set and R is an equivalence relation (generated from an attribute-value system), is nothing but a Kripke frame for the logic system **S5** [19, p. 58]. The lower and upper approximations can be viewed as operations that interpret the necessity and possibility operators respectively. In the following years, various directions on the study of *logics* emerging out of rough set theory were proposed [2,3,25–28,37,39,42,32,1,10]. Among these directions are two major approaches: a formula is interpreted for one, as a set in an approximation space, and for the other, as a rough set with respect to an approximation space. Nonetheless, in both approaches, the structure remains Pawlakian, i.e., it is a set with an equivalence relation or equivalently a partition.

Pawlakian rough set theory has been extended to various kinds of rough sets, e.g., similarity relation based rough sets [36,21], arbitrary binary relation based rough sets [23,43–45,48], and covering-based rough sets [8,9,31,47]. In some cases, only a fragment of logical studies connected with various rough set models is indicated in the literature [28,2,37,4,7,20,11,5,30,15,14,38,24,40,16,17,22] mostly from algebraic perspective. We shall work in the domain of covering-based rough sets. A covering $\mathscr{C}$ of a non-empty set X is a collection $\{C_i \mid i \in I\}$ of subsets of X such that $\bigcup \mathscr{C} = \bigcup_{i \in I} C_i = X$. The generalization to a covering from partition was a natural outcome both in theoretical and practical respects. Various forms of covering have been proposed, and they can be found in some survey works [33,34,46].

What has been lacking so far is the study on the logical systems that may have set models in the domain of covering-based rough sets. However, a preliminary work has recently been published toward this direction [35], where some questions have been raised and some conjectures have been made. The present investigation in some senses contributes in that direction but it will be observed that covering based rough sets lead to many deeper issues of modal logics. To

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understand these issues, it is necessary to be familiar with various types of lower and upper approximations of a set. We depend on [35] in this respect. The covering-based rough set systems that are considered in this paper rendering semantics to some modal logic systems are C_2, C_5, P_1 and C_4 in [35]. Definitions of the corresponding lower and upper approximations will be given in Definition 2.3. It should, however, be mentioned that the covering-based approximations presented in [35] are not exhaustive.

As is clear from [35, Table 3], each rough set system defined by a pair of lower/upper approximations could be the model of some modal logic system. To make this paper self-contained as much as possible, we reproduce a relevant portion of the table along with an explanation in Section 2.2. It is likely that some modal system hitherto unknown may emerge from some rough set semantics represented in various columns of the table. In this paper, however, we shall first concentrate on the conjectures that have been made in [35]. The first conjecture that the C_2 and C_5 semantics (cf. Definition 2.3) give rise to the same modal system which is at least **S4** and not **S5** is answered, and furthermore, we prove that the system is exactly the modal logic **S4**. Using our technique of proving completeness, i.e., reduction of the covering semantics to Kripke semantics, it is gratifying to see that the temporal logic **S4***t* also admits a covering-based semantics. The second conjecture that the P_1 and C_4 semantics give rise to the same modal system which is at least **KTB** and not **S5** is also answered, and we prove that the system is exactly the modal logic **KTB**.

The present paper is organized as follows. Preliminaries on Kripke semantics and covering semantics for modal logic will be given in Section 2. Section 3 presents a proof of the first conjecture that the C_2 and C_5 semantics give rise to the same modal system **S4**, and the covering semantics for the temporal logic **S4***t* is developed. Similarly, Section 4 presents a proof of the second conjecture that the P_1 and C_4 semantics give rise to the same modal logic **KTB**. Section 5 contains some concluding remarks and future directions of our research.

2. Preliminaries

The language of modal logic $\mathcal{L}_{\mathrm{ML}}$ consists of a denumerable set of propositional variables Prop, propositional connectives $\neg$ (negation), $\vee$ (disjunction) and the unary modality $\Box$. The set of all modal formulas $\mathcal{L}_{\mathrm{ML}}$ is defined inductively by the following rule:

$$\mathcal{L}_{\mathrm{ML}} \ni \alpha ::= p \mid \neg\alpha \mid (\alpha \vee \alpha) \mid \Box\alpha, \text{ where } p \in \mathsf{Prop}.$$

Other propositional connectives $\wedge$ (conjunction), $\to$ (implication) and $\leftrightarrow$ (equivalence) are defined as usual. The dual of $\Box$ is defined by $\Diamond\alpha := \neg\Box\neg\alpha$.

2.1. Kripke semantics

A Kripke frame is a pair $\mathcal{F} = (X, R)$ where X is a non-empty set of worlds and R is a binary relation on X — the accessibility relation. A frame $\mathcal{F} = (X, R)$ is said to be (i) *reflexive* if $\forall x \in X(xRx)$; (ii) *transitive* if $\forall x, y, z \in X(xRy \wedge yRz \to xRz)$; and (iii) *symmetric* if $\forall x, y \in X(xRy \to yRx)$.

A Kripke model is a triple $\mathfrak{M} = (X, R, V)$ where (X, R) is a Kripke frame and $V : \mathsf{Prop} \to \mathcal{P}(X)$ is a valuation function from Prop to the powerset of X. A reflexive and transitive Kripke frame (respectively model) is called an *S*4*-frame* (respectively *S*4-model).

Given a Kripke frame $\mathcal{F} = (X, R)$, we define the operation $\Box_R : \mathcal{P}(X) \to \mathcal{P}(X)$ on the powerset $\mathcal{P}(X)$ by setting

$$\Box_R A = \{x \in X \mid R(x) \subseteq A\}$$

where $R(x) = \{y \in X \mid xRy\}$. The dual operation of $\Box_R$ is defined by $\Diamond_R A = (\Box_R A^c)^c = \{x \in X \mid R(x) \cap A \neq \emptyset\}$, where $(.)^c$ is the complement operation.

Definition 2.1. The truth set $[\![\alpha]\!]_{\mathcal{M}}$ of a modal formula $\alpha \in \mathcal{L}_{\mathrm{ML}}$ in a Kripke model $\mathcal{M} = (X, R, V)$ is defined inductively by

$$[\![p]\!]_{\mathcal{M}} = V(p)$$
$$[\![\neg\alpha]\!]_{\mathcal{M}} = ([\![\alpha]\!]_{\mathcal{M}})^c$$
$$[\![\alpha \vee \beta]\!]_{\mathcal{M}} = [\![\alpha]\!]_{\mathcal{M}} \cup [\![\beta]\!]_{\mathcal{M}}$$
$$[\![\Box\alpha]\!]_{\mathcal{M}} = \Box_R [\![\alpha]\!]_{\mathcal{M}}$$

A formula α is true (or satisfied) at x in $\mathcal{M}$ (notation: $\mathcal{M}, x \models_K \alpha$, where the subscript K means 'Kripke') if $x \in [\![\alpha]\!]_{\mathcal{M}}$. A formula α is true in $\mathcal{M}$ (notation: $\mathcal{M} \models_K \alpha$) if $[\![\alpha]\!]_{\mathcal{M}} = X$. A formula α is *valid* at $x \in X$ in a Kripke frame $\mathcal{F} = (X, R)$ (notation: $\mathcal{F}, x \models_K \alpha$) if $x \in [\![\alpha]\!]_{\mathcal{F},V}$ for any valuation V in $\mathcal{F}$. A formula α is *valid* in $\mathcal{F}$ (notation: $\mathcal{F} \models_K \alpha$) if $\mathcal{F}, x \models_K \alpha$ for all $x \in X$.

The minimal normal modal system **K** consists of the following axiom schemata and inference rules:

- (Tau) All instances of classical propositional tautologies.
- (K) $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$.