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2-Coherent and 2-convex conditional lower previsions

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ABSTRACT

In this paper we explore relaxations of (Williams) coherent and convex conditional previsions that form the families of n-coherent and n-convex conditional previsions, at the varying of n. We investigate which such previsions are the most general one may reasonably consider, suggesting (centered) 2-convex or, if positive homogeneity and conjugacy is needed, 2-coherent lower previsions. Basic properties of these previsions are studied. In particular, we prove that they satisfy the Generalised Bayes Rule and always have a 2-convex or, respectively, 2-coherent natural extension. The role of these extensions is analogous to that of the natural extension for coherent lower previsions. On the contrary, n-convex and n-coherent previsions with $n \geq 3$ either are convex or coherent themselves or have no extension of the same type on large enough sets. Among the uncertainty concepts that can be modelled by 2-convexity, we discuss generalisations of capacities and niveloids to a conditional framework and show that the well-known risk measure Value-at-Risk only guarantees to be centered 2-convex. In the final part, we determine the rationality requirements of 2-convexity and 2-coherence from a desirability perspective, emphasising how they weaken those of (Williams) coherence.

1. Introduction

In his influential book *Statistical Reasoning with Imprecise Probabilities* [21], P. Walley developed a behavioural approach to *imprecise* probabilities (and previsions) extending de Finetti's [5] interpretation of coherent *precise* previsions. Operationally, this was achieved through a relaxation of de Finetti's betting scheme.

In fact, following de Finetti, P is a coherent precise prevision on a set $\mathcal{S}$ of gambles if and only if for all $m, n \in \mathbb{N}_0$, $s_1, \ldots, s_m, r_1, \ldots, r_n \geq 0$, $X_1, \ldots, X_m, Y_1, \ldots, Y_n \in \mathcal{S}$, defining $G = \sum_{i=1}^{m} s_i(X_i - P(X_i)) - \sum_{j=1}^{n} r_j(Y_j - P(Y_j))$, it holds that $\sup G \geq 0$. The terms $s_i(X_i - P(X_i))$, $-r_j(Y_j - P(Y_j))$ are proportional (with coefficients or *stakes* s_i, r_j) to the *gains* arising from, respectively, buying X_i at $P(X_i)$ or selling Y_j at $P(Y_j)$. A coherent lower prevision $\underline{P}$ on $\mathcal{S}$ may be defined in a similar way, just restricting n to belong to $\{0, 1\}$. This means that the betting scheme is modified to allow selling at most one gamble. Several other betting scheme variants have been investigated in the literature, either extending coherence for lower previsions (conditional lower previsions) or weakening it (previsions that are convex, or avoid sure loss). In particular, a convex lower prevision is defined introducing a convexity constraint $n = 1, \sum_{i=1}^{m} s_i = r_1 = 1$ in the betting scheme. In [21, Appendix B] n-coherent previsions are studied, as a different relaxation of coherence.

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In this paper, we explore further variations of the behavioural approach/betting scheme: n-coherent and n-convex conditional lower previsions, formally defined later on as generalisations of the n-coherent (unconditional) previsions in [21]. Our major aims are:

a) to explore the flexibility of the behavioural approach and its capability to encompass different uncertainty models;
b) to point out which are the basic axioms/properties of coherence which hold even for much looser consistency concepts.

Referring to b) and with a view towards the utmost generality, we shall mainly concentrate on the extreme quantitative models that can be incorporated into a (modified) behavioural approach. This does not imply that these models should be regarded as preferable to coherent lower previsions. On the contrary they will not, as far as certain questions are concerned. For instance, inferences will typically be rather vague. However, it is interesting and somehow surprising to detect that certain properties like the Generalised Bayes Rule must hold even for such models, or that they can be approached in terms of desirability.

N-coherence and n-convexity may be naturally seen as relaxations of, respectively, (Williams) coherence and convexity. These and other preliminary concepts are recalled in Section 2. Starting from the weakest reasonably sound consistency concepts, we explore 2-convex lower previsions in Section 3. In Section 3.1 we characterise them by means of axioms, on a special set of conditional gambles generalising a linear space and termed $\mathcal{D}_{LIN}$ (Definition 2, Proposition 2). Interestingly, it turns out that n-convexity with $n \geq 3$ and convexity are equivalent on $\mathcal{D}_{LIN}$. 2-Convex previsions display some drawbacks: in Section 3.2, it is shown that a 2-convex natural extension may be defined and its properties are discussed, but its finiteness is not guaranteed. Moreover, as detailed in Section 3.3, the property of internality may fail (with some limitations, for instance lack of internality cannot be two-sided); agreement with conditional implication (the Goodman–Nguyen relation) is not guaranteed either. In Section 3.4, we show that the special subset of centered 2-convex previsions is not affected by these problems. In Section 4, 2-coherent lower previsions are discussed and characterised on $\mathcal{D}_{LIN}$ (Proposition 9). We compare 2-coherence and n-coherence in Section 4.1: again, n-coherence ($n \geq 3$) and coherence are equivalent on $\mathcal{D}_{LIN}$. N-coherent previsions ($n \geq 3$) defined on a generic set of gambles $\mathcal{S}$ have no n-coherent extension on sufficiently large supersets whenever the equivalence does not hold already on $\mathcal{S}$. We show also that 2-coherence should be preferred to 2-convexity when positive homogeneity and conjugacy are required. The 2-coherent natural extension is introduced and studied in Section 4.2. 2-Coherent lower previsions always have it. The extent of the Generalised Bayes Rule for 2-coherent lower previsions is discussed in Section 4.3. Models that can be accommodated into the framework of 2-convexity or 2-coherence, but not of coherence, are presented in Section 5. We focus on how 2-convexity can motivate defining conditional versions of capacities and niveloids, and on the consistency properties of Value-at-Risk, a well-known risk measure which is centered 2-convex, but may even fail to be 2-coherent. In Section 6 we analyse 2-convexity and 2-coherence in a desirability approach. Generalising prior work by Williams [22,23] for coherence, we focus on the correspondence between these previsions and sets of desirable gambles, and on establishing the ensuing desirability rules. The major differences with the rules for Williams coherence are pointed out in the comments following Propositions 17 and 20. Section 7 concludes the paper. An earlier presentation of the topics in this paper, less extended and without proofs, was delivered at the ISIPTA'15 Symposium [16].

2. Preliminaries

The starting points for our investigation are the known consistency concepts of coherent and convex lower conditional prevision [13,14,22,23]. They both refer to an arbitrary (non-empty) set $\mathcal{D}$ of conditional gambles, that is of conditional bounded random variables. We denote by $X|B$ a generic conditional gamble, where X is a gamble and B is a non-impossible event ($B \neq \varnothing$). It is understood here that $X : \mathbb{P} \to \mathbb{R}$ is defined on an underlying partition $\mathbb{P}$ of atomic events ω, and that B belongs to the powerset of $\mathbb{P}$. Therefore, any $\omega \in \mathbb{P}$ implies either B or its negation $\neg B$ (in words, knowing that ω is true determines the truth value of B, i.e. B is known to be either true or false). Given B, the conditional partition $\mathbb{P}|B$ is formed by the conditional events $\omega|B$, such that ω implies B (implies that B is true) and $X|B : \mathbb{P}|B \to \mathbb{R}$ is such that $X|B(\omega|B) = X(\omega)$, $\forall \omega|B \in \mathbb{P}|B$. Because of this equality, several computations regarding $X|B$ can be performed by means of the restriction of X on B. In particular, it is useful for the sequel to recall that $\sup(X|B) = \sup_B X = \sup\{X(\omega) : \omega \in \mathbb{P}, \omega \Rightarrow B\}$, and $\inf(X|B) = \inf_B X = \inf\{X(\omega) : \omega \in \mathbb{P}, \omega \Rightarrow B\}$.

As a special case, letting Ω be the sure event, we have that $X|\Omega = X$ is an unconditional gamble. Further, $A|B$ is a conditional event if A is an event (or its indicator I_A – we shall generally employ the same notation A for both).

As customary, without further qualifications, a lower prevision $\underline{P}$ is a map from $\mathcal{D}$ into the real line, $\underline{P} : \mathcal{D} \to \mathbb{R}$. However, a lower prevision is often interpreted as a supremum buying price [21]. For instance, if a subject assigns $\underline{P}(X|B)$ to $X|B$, he is willing to buy X, conditional on B occurring, at any price lower than $\underline{P}(X|B)$. Referring to this behavioural interpretation, the following Definitions 1, 3, 5 require different degrees of consistency for $\underline{P}$, according to whether certain gains depending on $\underline{P}$ avoid losses bounded away from 0. They differ as to the buying and selling constraints they impose.

Definition 1. Let $\underline{P} : \mathcal{D} \to \mathbb{R}$ be given.