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Inconsistency indicator maps on groups for pairwise comparisons ☆

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This study presents an abelian group approach to analyzing inconsistency in pairwise comparisons. A notion of an inconsistency indicator map on a group, taking values in an abelian linearly ordered group, is introduced. For it, metrics and generalized metrics are utilized. Every inconsistency indicator map generates both a metric on a group and an inconsistency indicator of an arbitrary pairwise comparisons matrix over the group.

1. Introduction

The first documented use of pairwise comparisons (PC) is attributed to Ramon Llull, a 13th-century mystic and philosopher. Thurstone applied pairwise comparisons to experimental psychology and delivered the first formal introduction of pairwise comparisons in the form of "the law of comparative judgement" in [17]. A variation of this law is known as the BTL (Bradley–Terry–Luce) model (cf. [3]). A number of controversial customized pairwise comparisons have been considered in numerous studies. However, we do not intend to support any customization here. Amongst many others, Saaty's seminal work [16] had a considerable impact on the pairwise comparisons (PC) research. The authors' position is that the influence of [16] on the pairwise comparisons research should be acknowledged despite serious controversies generated by it. This work provides a more refined alo-group perspective on inconsistency in pairwise comparisons originated in [2] where deficiencies exist and some (but not all) are addressed in this study.

All measurements (physical or not) are related to pairwise comparisons. For example, when we say that the distance between stars A and B is 2.71 light years, we simply compare the unit of distance (in this case, one light year) to the distance between A and B. Pairwise comparisons are of particular use where a unit of measurement cannot be defined and it is so for most subjective assessments. For example, public safety or environmental pollution lacks a unit (or a "yard

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stick") for the measurement, however, it is still necessary to measure it by comparing them to each other and expressing it by a ratio stored in what we call a PC matrix.

A triad (x, y, z) of entries of PC matrix, expressing ratios of three entities, is called consistent if $xz = y$. This is a case of *multiplicative consistency* expressing ratios of compared entities. Another approach to the consistency of triads of numbers is additive. Namely, a triad x, y, z of numbers is called *additively consistent* if $x + z = y$. Additive consistency is relevant to the question: "by how much x is larger (or more important) than y". In most studies about PC, the multiplicative consistency has been mostly considered (e.g., [1]). Additive PC have been analyzed in [19] recently. Differences between quality values instead of ratios appear in BTL model. In [2], both multiplicative and additive pairwise comparisons were unified to the comparisons of elements of abelian linearly ordered groups (in abbreviation; alo-groups). A consistency index of triads of elements of an alo-group was defined in [2] (for a single triad, defined in [8] of 1993). An inconsistency indicator for triads, introduced in [8] for the multiplicative case in 1993 for a single triad was extended in [4]. The analysis of Koczkodaj's inconsistency indicator is included in [1] and [11]. An axiomatization of inconsistency in pairwise comparisons was proposed in [11] (with some minor deficiencies being currently corrected).

The main aim of this work is to introduce and investigate a new general concept of an inconsistency indicator map on a group, strictly relevant to a generalized metric which takes values in an alo-group. The inconsistency indicator maps induce inconsistency indicators of pairwise comparisons matrices.

Axioms of logic, rules of deduction and system ZF are assumed in this work. The notation and terminology of [12] is followed. In particular, ω is the set of all non-negative integers (of von Neumann), $0 = \emptyset$, $1 = \{0\}$ and $n + 1 = n \cup \{n\}$ for each $n \in \omega$ (cf. [14]).

2. A PC matrix over a group

Let $X = \langle X, \cdot \rangle$ be a group. We denote by 1_X or, for simplicity, by 1 the unit element of X. For $a \in X$, the symmetric (inverse) element of a in X is denoted by a^{-1}.

Let K be a non-void finite set. A $K \times K$ matrix $A = [a_{i,j}]$ over X is a mapping $A : K \times K \to X$ such that $a_{i,j} = A(i, j)$ for each pair $\langle i, j\rangle \in K \times K$. There exists $n \in \omega$ such that the sets n and K are equipollent. Let $\psi : n \to K$ be a bijection. We can define an $n \times n$ matrix A_ψ over X by $A_\psi(i, j) = A(\psi(i), \psi(j))$ for each $\langle i, j\rangle \in n \times n$. Therefore, without loss of generality, we can concentrate on $n \times n$-matrices where $n \in \omega \setminus \{0\} = \{1, 2, \ldots\}$.

Definition 2.1. For $n \in \omega \setminus \{0\}$, let $A = [a_{i,j}]$ be an $n \times n$ matrix such that $a_{i,j} \in X$ for all $i, j \in n$. We say that:

(i) the matrix A is a *pairwise comparisons matrix* (in abbreviation a *PC matrix*) *over the group* X if $a_{i,i} = 1_X$ and $a_{i,j} = a_{j,i}^{-1}$ for all $i, j \in n$;
(ii) the matrix A is a *consistent matrix over the group* X if $a_{i,k} \cdot a_{k,j} = a_{i,j}$ for all $i, j, k \in n$.

Remark 2.2. If $\cdot$ is the standard multiplication of positive real numbers, we call the ordered pair $\mathbb{R}_+ = \langle (0; +\infty), \cdot \rangle$ the *standard multiplicative group of positive real numbers*. The notion of a PC matrix over this group coincides with the usual notion of a PC matrix used by many PC researchers. Pairwise comparisons matrices over a group equipped with a linear order were also considered in [2].

Fact 2.3. *Every consistent $n \times n$ matrix $A = [a_{i,j}]$ over a group X is a PC matrix over X.*

3. An inconsistency indicator map on a group

To formulate a definition of an inconsistency indicator map, we will use the following notion of an alo-group investigated in [2] and, for example, also in [7]:

Definition 3.1. An *abelian linearly ordered group* (abbreviated to "alo-group") is an ordered pair $\langle \langle G, \odot \rangle, \leq \rangle$ where $\langle G, \odot \rangle$ is an abelian group, while $\leq$ is a linear order on G such that if $a, b, c \in G$ and $a \leq b$, then $a \odot c \leq b \odot c$.

Distance functions taking values in alo-groups were considered, for instance, in [2] and [7]. We modify Definition 3.2 of [2], by dropping its first condition $d(a, b) \geq e$ and changing G to X in the domain of d, to the following:

Definition 3.2. Let $\mathcal{G} = \langle \langle G, \odot \rangle, \leq \rangle$ be an alo-group. Let 1_G be the neutral element of $\langle G, \odot \rangle$. A *$\mathcal{G}$-metric* or a *$\mathcal{G}$-distance* on a set X is a function $d : X^2 \to G$ such that, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 1_G \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, z) \odot d(z, y)$.