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Generalized fuzzy rough approximation operators determined by fuzzy implicators

Wei-Zhi Wu [a,*], Yee Leung [b,c], Ming-Wen Shao [d]

[a] *School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan, Zhejiang 316004, PR China*
[b] *Department of Geography and Resource Management, The Chinese University of Hong Kong, Hong Kong*
[c] *Institute of Environment, Energy and Sustainability, The Chinese University of Hong Kong, Hong Kong*
[d] *Computer Engineering Institute, Qingdao Technological University, Qingdao, Shandong 266520, PR China*

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ABSTRACT

In this paper, a general framework for the study of dual fuzzy rough approximation operators determined by a fuzzy implication operator $\mathcal{I}$ in infinite universes of discourse is investigated. Lower and upper approximations of fuzzy sets with respect to a fuzzy approximation space in infinite universes of discourse are first introduced. Properties of $\mathcal{I}$-fuzzy rough approximation operators are then examined. An operator-oriented characterization of fuzzy rough sets is further proposed, that is, $\mathcal{I}$-fuzzy rough approximation operators are defined by axioms. Different axiom sets of lower and upper $\mathcal{I}$-fuzzy set-theoretic operators guarantee the existence of different types of fuzzy relations which produce the same operators. Finally, a comparative study of $\mathcal{I}$-fuzzy rough sets with fuzzy topological spaces is presented. It is proved that there exists a one-to-one correspondence between the set of all reflexive and $\mathcal{T}$-transitive fuzzy approximation spaces and the set of all fuzzy Alexandrov spaces such that the lower and upper $\mathcal{I}$-fuzzy rough approximation operators in a fuzzy approximation space are, respectively, the fuzzy interior and closure operators in a fuzzy topological space.

1. Introduction

The theory of rough sets was originally proposed by Pawlak [36,37] as a formal tool for modeling and processing incomplete information. The basic structure of rough set theory is an approximation space consisting of a universe of discourse and a binary relation imposed on it. Based on the approximation space, the notions of lower and upper approximation operators can be constructed. Using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules.

In the classical Pawlak's rough set model [37], an equivalence relation in the approximation space is a primitive notion. This model is very useful in the analysis of data in complete information systems/tables [37]. However, the requirement of an equivalence relation in Pawlak's rough set model seems to be a very restrictive condition that may limit the applications of the rough set model. Thus one of the main directions of research in rough set theory is naturally the generalization of the Pawlak rough set approximations. Many authors have generalized the notion of rough approximation operators by using non-equivalence binary relations, and the non-equivalence relation-based rough set models have been used in reasoning and knowledge acquisition with data sets presented as incomplete information tables.

* Corresponding author. Tel.: +86 580 8180928; fax: +86 580 2550029.
E-mail addresses: wuwz@zjou.edu.cn (W.-Z. Wu), yeeleung@cuhk.edu.hk (Y. Leung), mingwenshao@gmail.com (M.-W. Shao).

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Rough sets can also be generalized to the fuzzy environment, and the results of approximations of fuzzy sets in a crisp approximation space, and approximations of crisp sets or fuzzy sets in a fuzzy approximation space, are, respectively, called rough fuzzy sets and fuzzy rough sets [4,9,11,27,29,31–33,35,39,46–48,51,52,54,55]. The rough fuzzy set model can be used to deal with attribute reduction in information systems with fuzzy decision while the fuzzy rough set model can be employed in reasoning and knowledge acquisition with decision tables with real valued conditional attributes or quantitative data (see e.g. [6,16–22,25,44,56,57]).

By using the triangular norm Min and its dual conorm Max, Dubois and Prade [11] introduced the lower and upper approximations of fuzzy sets with respect to a Pawlak approximation space and a fuzzy similarity relation to obtain the extended notions called rough fuzzy sets and fuzzy rough sets, respectively. Based on arbitrary approximation spaces both in crisp and fuzzy cases, Wu et al. [51,52] defined generalized rough fuzzy sets and fuzzy rough sets determined by the triangular norm Min and its dual conorm Max, in which the lower and upper approximation operators are dual with each other. Notice that Min and Max are special triangular norm and conorm, since there are many triangular norms [23] used in the fuzzy set theory, Morsi and Yakout [33], and Radzikowska and Kerre [39] developed a broad family of the so-called $(\mathcal{I},\mathcal{T})$-fuzzy rough sets which are determined by an implicator (implication operator) $\mathcal{I}$ and a triangular norm $\mathcal{T}$ on the unit interval $[0,1]$, but their studies were defined by $\mathcal{T}$-similarity fuzzy relations which are equivalence crisp relations in the degenerated case. In [48], Wu et al. explored the concept of $(\mathcal{I},\mathcal{T})$-fuzzy rough sets defined by arbitrary fuzzy relations and characterized essential properties of the fuzzy approximation operators. In [40], Radzikowska and Kerre introduced L-fuzzy rough sets as a generalization of the notion of rough sets determined by a residuated lattice L.

However, the lower and upper fuzzy rough approximation operators in [33,39,40,48], in general, are not dual with each other. We know that the dual properties of lower and upper approximation operators are of particular importance in the analysis of mathematical structures in rough set theory. The dual pairs of lower and upper approximation operators in the rough set theory are strongly related to the interior and closure operators in topological space, the necessity (box) and possibility (diamond) operators in modal logic, and the belief and plausibility functions in the Dempster–Shafer theory of evidence. By introducing the dual operator of a residual fuzzy implication operator (also called R-implicator), Mi and Zhang presented the constructive definition of upper fuzzy approximation operator (also called σ-upper approximation operator [6]) which is dual to the lower fuzzy approximation operator (also called θ-lower approximation operator [6]) defined by Morsi and Yakout [33]. Determined by a general type of triangular norm $\mathcal{T}$ with its dual triangular conorm $\mathcal{S}$, Mi et al. [31] and Wu [46] recently investigated different pairs of dual fuzzy rough approximation operators (called $\mathcal{T}$-upper approximation operator and $\mathcal{S}$-lower approximation operator [6]).

It is well known that there are a lot of implicators [41] which have been widely used in fuzzy sets research. It should be noted that fuzzy inference results often depend upon the choice of the implicator and choice of the triangular norm. For analyzing uncertainty in complicated fuzzy systems, lower and upper fuzzy rough approximations defined by arbitrary implicators in rough set theory need to be developed. Determined by an arbitrary implicator $\mathcal{I}$, the so-called $\mathcal{I}$-lower fuzzy approximation operator has been proposed and investigated [39,48], and the θ-lower approximation operator and $\mathcal{S}$-lower approximation operator can be regarded as special cases of $\mathcal{I}$-lower fuzzy approximation operator when taking the particular R-implicator and S-implicator, respectively [48]. However, the constructive definition with its mathematical structures of upper fuzzy approximation operator which is dual to the $\mathcal{I}$-lower fuzzy approximation operator in an arbitrary fuzzy approximation space have not been studied.

In this paper, we will present a general framework for the study of dual pair of lower and upper fuzzy rough approximation operators determined by a general fuzzy implicator $\mathcal{I}$ in infinite universes of discourse. In the next section, we review some basic notions and results of fuzzy logical operators, fuzzy relations, and generalized rough approximation operators. In Section 3, based on an arbitrary fuzzy relation, a pair of lower and upper $\mathcal{I}$-fuzzy rough approximation operators are defined, and their properties are then examined. On the other hand, properties of $\mathcal{I}$-fuzzy rough approximation operators can be used to characterized special types of fuzzy relations. In Section 4, $\mathcal{I}$-fuzzy rough sets are axiomatized by abstract operators. Various classes of $\mathcal{I}$-fuzzy approximation operators are characterized by different sets of axioms, and the axiom set of the $\mathcal{I}$-fuzzy rough approximation operators guarantee the existence of certain types of fuzzy relations producing the same operators. We further establish the relationship between $\mathcal{I}$-fuzzy rough approximation operators and fuzzy topological spaces in Section 5. We then conclude the paper with a summary in Section 6.

2. Preliminaries

In this section we recall some basic notions and previous results which will be used in the later parts of this paper.

2.1. Fuzzy logical operators

A *triangular norm* [23], or t-norm in short, is an increasing, associative and commutative mapping $\mathcal{T}: I^2 \to I$ (where $I=[0,1]$ is the unit interval) that satisfies the boundary condition: for all $\alpha \in I$, $\mathcal{T}(\alpha,1)=\alpha$. The most popular continuous t-norms are:

- the standard min operator $\mathcal{T}_{\mathrm{M}}(\alpha,\beta)=\min\{\alpha,\beta\}$ (the largest t-norm [24]),
- the algebraic product $\mathcal{T}_{\mathrm{P}}(\alpha,\beta)=\alpha * \beta$,