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On fuzzy implications: An axiomatic approach

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ABSTRACT

Fuzzy operations acting on entire fuzzy sets with the stress on fuzzy implications are discussed and studied. In the case of binary operations, the input fuzzy sets are fuzzy subsets of possibly different universal spaces X and Y, and the output fuzzy set is a fuzzy subset of the Cartesian product $X \times Y$. The standard approach to fuzzy operations is based on functions acting on $[0,1]$, and then these fuzzy operations are called functionally expressible. We give a characterization of functionally expressible fuzzy implications (and other fuzzy operations), and include several examples of fuzzy operations which are not functionally expressible.

1. Introduction

Since the introduction of Zadeh's theory of fuzzy sets in 1965, there has been a rapid growth of interest in the science of approximate reasoning and its applications to modeling imprecise systems. One of the most important differences between the many models of approximate reasoning is in the variation in the definition of implication, where for propositions A and B, "A implies B" is written $A \to B$ and may be interpreted linguistically by "If A, then B". In ordinary binary logic, the implication function is simply expressed by the truth table of $\neg A \vee B$ (i.e. NOT A, OR B) and this same function has been used to define a fuzzy law of implication using the closed interval $[0,1]$ of truth values instead of the classical two values $\{0,1\}$: $I(a,b) = \max\{1-a, b\}$ where a, b are in $[0,1]$.

In this paper we distinguish between implications on fuzzy propositions, that we call fuzzy implications, and functions that can be used to build fuzzy implications, that we call implication functions. The usual way to construct/modelize a fuzzy implication is from a numerical function (implication function) $I : [0,1] \times [0,1] \to [0,1]$ and defining

$$(A \to B)(x, y) = I\big(A(x), B(y)\big)$$

where x, y are elements of the (possibly different) domains X, Y, A, B are the corresponding fuzzy propositions (or fuzzy sets on X and Y respectively) and $A(x)$, $B(y)$ are the truth values of A and B at the points x, y respectively (or equivalently, the membership values of x, y to the fuzzy sets A an B). Note that, once selected the implication function I, we assume with this model that the value $(A \to B)(x, y)$ depends only on the values taken by A and B at the points x and y. In this case, we say that the fuzzy implication $\to$ is functionally expressible from the implication function I. Clearly, properties

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of $\rightarrow$ and I are moved to one another. In this paper we consider implication functions satisfying the following basic conditions: decreasing in the first variable, increasing in the second variable and with the following boundary behavior: $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$ (I must be an extension of the classical implication). In classical logic, where $a, b \in \{0,1\}$, the classical implication can be described in several distinct functional forms:

$$I(a,b) = \max\{1-a, b\} = 1-a+ab = \min\{1, 1-a+b\} = \text{etc.},$$

but while these forms are equivalent in the classical case, their extensions to fuzzy logic are not equivalent and result in distinct classes of implication functions. A recent and complete monograph on this kind of functions is [1]. Recently, many other works have appeared dealing with these implication functions. Some of the topics in this field are characterizations, generalizations and intersections of known families [13–15,20], the study of additional properties [2,18] and the applications of implication functions [5,17].

However, in many cases fuzzy set operations depend on the complete information about the involved fuzzy sets and thus, they need not be functionally expressible. This is the main reason for the interest in non-functionally expressible operations and, consequently, for an axiomatic definition of these operations. This interest was already shown by E.P. Klement in [10]. For some further studies in this direction see, for example [16,19,21]. Note that in several works, L-fuzzy sets proposed by Goguen [7] are considered, with axiomatically characterized operations on a bounded lattice L. Obviously, then the problem of clarifying the structure and properties of discussed operations is of interest, especially in the cases when $L = [0,1]^X$ is a system of all fuzzy subsets of a fixed space X (compare, for example, type-2 fuzzy sets [9]). In our contribution, one step forward is aimed to be done. Namely, as already mentioned, we look on implications as a connective acting on two, possibly different, systems of fuzzy sets $[0,1]^X$ and $[0,1]^Y$, with an output in $[0,1]^{X\times Y}$. We want mainly deal with non-functionally expressible fuzzy implications. In this way, after introducing some preliminaries in Section 2, we present an axiomatic definition of fuzzy implications in Section 3 and characterize those which are functionally expressible. Then we introduce a method to construct fuzzy implications from two implication functions and an aggregation function having 0 and 1 as left-absorbent elements, and characterize those which are non-functionally expressible. A class of such aggregation functions can be obtained slightly modifying a t-norm and a t-conorm.

2. Preliminaries

We will suppose the reader to be familiar with the theory of t-norms and t-conorms (all necessary results and notations can be found in [11]), with the theory on implication functions (see [1]) and with the theory of aggregation functions (all necessary results and notations can be found in [3] and more deeply in [4] and [8]). To make this work self-contained, we recall here some of the concepts and results employed in the rest of the paper.

First of all, we recall the concept of a fuzzy negation, a logical operator involved many times in the definition of some classes of fuzzy implications.

Definition 1. (See Definition 1.1 in [6].) A decreasing function $N : [0,1] \rightarrow [0,1]$ is called a *fuzzy negation*, if $N(0) = 1$, $N(1) = 0$. A fuzzy negation N is called

(i) *strict*, if it is strictly decreasing and continuous;
(ii) *strong*, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Until now, the role of the binary implication in crisp logic has been performed in fuzzy logic by implication functions. The most accepted definition of an implication function is the following one:

Definition 2. (See Definition 1.15 in [6].) A binary operator $I : [0,1]^2 \rightarrow [0,1]$ is said to be an *implication function* if it satisfies:

(I1) $I(x,z) \geqslant I(y,z)$ when $x \leqslant y$, for all $z \in [0,1]$.
(I2) $I(x,y) \leqslant I(x,z)$ when $y \leqslant z$, for all $x \in [0,1]$.
(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition.

Definition 3. We will say that an implication function $I : [0,1]^2 \rightarrow [0,1]$ is *trivial* when it takes only the values 0 and 1. That is, when

$$I(x,y) \in \{0,1\} \quad \text{for all } x, y \in [0,1].$$