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Ordinally equivalent data: A measurement-theoretic look at formal concept analysis of fuzzy attributes

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ABSTRACT

We show that if two fuzzy relations, representing data tables with graded attributes, are ordinally equivalent then their concept lattices with respect to the Gödel operations on chains are (almost) isomorphic and that the assumption of Gödel operations is essential. We argue that measurement-theoretic results like this one are important for pragmatic reasons in relational data modeling and outline issues for future research.

1. Introduction and problem setting

A frequent objection to using degrees in representing vague terms such as "tall" can be articulated as follows. Why to assign the truth degree 0.764 to the proposition "John is tall"? Why not 0.682? This objection has a clear pragmatic aspect and suggests a fundamental problem in using truth degrees. The objection is found in various forms in the literature on vagueness, see e.g. [31, pp. 52–53] and also [13,18,19], and in many debates since the inception of fuzzy logic.

Whether and to what extent this objection, appealing as it is, indeed presents a problem, calls for close scrutiny. Presumably, one needs to look for answers pertaining to the usage of truth degrees in general as well as those that apply to particular models and applications. In our view, the issues involved are naturally looked at from the viewpoint of the theory of measurement. Measurement theory has been initiated by Stevens [33], in which the so-called ordinal, interval and ratio scales were recognized, and further developed in many publications within mathematical psychology, see e.g. [12,21,25,27,30].

In this paper, we examine some of the questions offered by the above considerations in a limited scope of a particular area, namely formal concept analysis (FCA), see [14,11]. Limited as it is, formal concept analysis encompasses rather general structures such as lattices, closure structures and operators, and Galois connections, hence the ramifications are broad. The basic problem we consider may be described as follows. Consider the following table, representing fuzzy relation I_1 between objects x_1, x_2 and x_3, and attributes $y_1, \ldots, y_4$.

I_1	y_1	y_2	y_3	y_4
x_1	1	0.9	0.8	1
x_2	0	1	0.5	0.5
x_3	0.8	0.8	0.2	0.1

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To what extent do the values of truth degrees, i.e. 1, 0.9, 0.8, etc. matter? What happens if we replace 0.8 by 0.7 in the three entries in the table? This question is important from a pragmatic viewpoint. Namely, when filling in the table, by a domain expert or a data analyst, one needs to know about the impact of the values and their relationships on further processing of the table. Since the basic structures utilized in FCA are concept lattices derived from such data tables, we are particularly interested in the impact on the structure of the concept lattice corresponding to the given data table.

In our previous work [1], later extended to the framework of general relational structures of first-order fuzzy logic [5], we showed then with an appropriately defined notion of similarity, the following claim can be proven: the degree of similarity of two data tables is less than or equal to the degree of similarity of the corresponding concept lattices, i.e. similar data tables lead to similar concept lattices. Hence, in a sense, the exact values of the truth degrees do not actually matter as far as the associated concept lattice is concerned.

In this paper, we examine a related but different issue. It consists in considering as essential the ordering of truth degrees, rather than the particular (numerical) values representing them. This view is implicitly present in describing fuzzy logic as a "logic of comparative truth". To make our point more concrete, consider as a simple example three propositions, φ_1, φ_2, and φ_3, and two truth valuations, e_1 and e_2, corresponding to two experts. Let $e_1(\varphi_1) = 0.2$, $e_1(\varphi_2) = 0.5$, $e_1(\varphi_3) = 0.9$, and $e_2(\varphi_1) = 0.15$, $e_2(\varphi_2) = 0.63$, $e_2(\varphi_3) = 0.8$. Even though the degrees assigned to the same proposition by the two experts are different, and one sometimes has $e_1(\varphi_i) < e_2(\varphi_i)$ and sometimes $e_1(\varphi_i) > e_2(\varphi_i)$, there is still an important kind of consistency of e_1 with e_2. Namely, for every pair φ_i and φ_j of propositions we have

$$e_1(\varphi_i) \leqslant e_1(\varphi_j) \quad \text{if and only if} \quad e_2(\varphi_i) \leqslant e_2(\varphi_j).$$

Similar kind of consistency in using degrees of membership was reported in experimental work on the psychology of concepts in the early 1970s [24,28,29]. Continuing with our example, one might call the expert assignments e_1 and e_2 ordinally equivalent and ask whether and under which conditions a further processing based on φ_1, φ_2, and φ_3 corresponding to the two truth valuations results in two consistent conclusions.

In this paper, we define the notion of ordinal equivalence for data tables with fuzzy attributes and prove that when using the Gödel logic connectives on linearly ordered sets of degrees, the concept lattices associated to ordinally equivalent data tables are almost isomorphic (see Remark 1) with the corresponding formal concepts pairwise ordinally equivalent. In addition, if the tables are even strongly ordinally equivalent, the concept lattices are isomorphic. We describe the isomorphisms and prove that the assumption of Gödel operations is essential. Results of this kind are important in addressing the issues regarding the significance of the values of truth degrees and the choice of fuzzy logic connectives in formal concept analysis as well as in a broader context of fuzzy logic models. The preliminary notions are surveyed in Section 2. Section 3 presents the results. We conclude the paper by a summary and a brief outline of future research issues.

2. Preliminaries

Structures of truth degrees As a scale of truth degrees we use a complete residuated lattice [15–17], i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); and $\otimes$ and $\rightarrow$ satisfy the adjointness property: $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm [16] with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel: ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leqslant b$, $a \rightarrow b = b$ else), Goguen (product): ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leqslant b$, $a \rightarrow b = \frac{b}{a}$ else). Namely, all other continuous t-norms are obtained as ordinal sums of these three [16,17]. Alternatively, we can take a finite subset $L \subseteq [0, 1]$ equipped with appropriate operations. Having $\mathbf{L}$ as the structure of truth degrees, we use the usual notions of fuzzy sets and fuzzy relations [2,16,34].

Formal concept analysis of data with fuzzy attributes Let X and Y be finite non-empty sets of objects and attributes, respectively, I be a fuzzy relation between X and Y. That is, $I : X \times Y \rightarrow L$ assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ to which the object x has the attribute y. The triplet $\langle X, Y, I\rangle$, called a *formal* $\mathbf{L}$*-context*, represents a data table, such as the one shown above, with rows and columns corresponding to objects and attributes, and table entries containing degrees $I(x, y)$.

For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^\uparrow \in L^Y$ and $B^\downarrow \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by

$$A^\uparrow(y) = \bigwedge_{x \in X} \big(A(x) \rightarrow I(x, y)\big) \quad \text{and} \quad B^\downarrow(x) = \bigwedge_{y \in Y} \big(B(y) \rightarrow I(x, y)\big).$$

Using basic rules of predicate fuzzy logic, $A^\uparrow(y)$ is the truth degree of "for each $x \in X$: if x belongs from A then x has y". Similarly for $B^\downarrow$. That is, $A^\uparrow$ is a fuzzy set of attributes common to all objects of A, and $B^\downarrow$ is a fuzzy set of objects sharing all attributes of B. The set

$$\mathcal{B}(X, Y, I) = \big\{\langle A, B\rangle \mid A^\uparrow = B,\ B^\downarrow = A\big\},$$