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Bounding the degree of solutions of differential equations

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ABSTRACT

We present an algorithmic strategy to compute an upper bound for the degree of the algebraic solutions of non-degenerate polynomial differential equations in dimension two.

1. Introduction

In 1878 Gaston Darboux published a paper (Darboux, 1878) on which both Poincaré (1891, p. 193) and Painlevé (1897, p. 217) bestowed the adjective “magistral”. In this paper, Darboux introduced a new method for finding a first integral of a differential equation defined by a homogeneous projective 1-form Ω; see section 2 for definitions. More precisely, Darboux showed that if Ω has degree n and more than $(n+1)(n+2)/2$ distinct algebraic solutions, then it has a first integral. Thus, in order to apply Darboux's method it is enough to find the algebraic solutions of a given polynomial differential equation.

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Drawing inspiration from Darboux's work, and from later work by Painlevé and Autonne, Poincaré published in 1891 a paper (Poincaré, 1891) devoted to the investigation of the rational first integrals of Ω. Both the numerator and the denominator of such an integral must be algebraic solutions of Ω. Poincaré points out that, in principle, this reduces the problem of finding a rational first integral to that of bounding the degrees of the algebraic solutions. At least that is so if, in Poincaré's words, one can "find some way of expressing, in the inequalities, that this integral is irreducible", by which he means that it cannot be written as the composition of a polynomial with a rational first integral.

After lying dormant for some decades, the work of Darboux was reworked in the language of modern algebraic geometry, and vastly generalized, by Jouanolou (1979). As a consequence of Jouanolou (1979, Proposition 4.1(ii), p. 126), one has that the degree of the smooth algebraic solutions of Ω cannot be greater than $n+1$. Moreover, it follows from Jouanolou (1979, Theoreme 3.3, p. 102) that Ω has infinitely many algebraic solutions if and only if it has a rational first integral. In particular, if Ω does not have a rational first integral, then there is an upper bound for the degree of its algebraic solutions. In the early 1980s, the problem of bounding the degrees of algebraic solutions also appeared in the work of Prelle and Singer on elementary first integrals of differential equations (Prelle and Singer, 1983, Problem D, p. 227).

The *Poincaré problem*, as the problem of bounding the algebraic solutions of polynomial vector fields came to be called, was explicitly stated in a paper of Cerveau and Lins Neto (1991, Theorem 1, p. 891). This paper also contains a generalization of Jouanolou's bound to nodal curves. From the algorithmic point of view, this result has the advantage that there are hypotheses on Ω that force its algebraic solutions to be nodal curves. Similar bounds were obtained by Walcher (2000) and Carnicer (1994). All these bounds require Ω to satisfy some extra hypothesis, in addition to the non-existence of a first integral.

In this paper we approach this problem from a different point of view. Instead of giving a formula for the required upper bound, we describe an algorithmic strategy that can be used to find such a bound for a given polynomial 1-form. This strategy is not guaranteed to succeed, and requires the foliation $\mathcal{F}$, induced by Ω on the complex projective plane, to be non-degenerate. However, it produces upper bounds for differential equations that are not covered by any of the previous results; for example, when the foliation has dicritical singularities; see section 5.

The algorithms are based on two index formulae proved by Moulin Ollagnier (2004), one of which is a version of the well-known Camacho–Sad Index Theorem (Camacho and Sad, 1982). The key point of Moulin-Ollagnier's formulae is that the indices may be written as linear combinations, with non-negative integer coefficients, of the characteristic ratios of the foliations at its singularities. The importance of characteristic ratios in the search for algebraic solutions of foliations goes back to Poincaré (1891, p. 39); since then it has been extensively used; see, for instance, Lins Neto (1988) and Chavarriga et al. (2005).

Throughout the paper we assume that the foliation $\mathcal{F}$ is non-degenerate and that it is defined over an *effective field* $K \subset \mathbb{C}$. In other words, K is a field for which the operations of addition, subtraction, multiplication, and division by non-zero elements can be implemented in a computer. Under these hypotheses, the index formulae give rise to a number of systems of diophantine equations of degrees 1 and 2, one of whose variables corresponds to the degree of any algebraic solutions the foliation may have. Thus, whenever all these systems have a finite number of solutions, we get an upper bound on the degree of the algebraic solutions of the foliation. The diophantine equations can be easily determined using the algorithms of section 4, where we also describe the strategy used to compute the desired upper bound. The theorems on which these algorithms are based are proved in section 3. The strategy of section 4 is applied to a number of examples in section 5. Since all our results are stated in terms of holomorphic foliations, we review, in section 2, the concepts and main results of this theory that are used in later sections. All the algorithms described in this paper were implemented for the base field $\mathbb{Q}$ using the computer algebra system Axiom (Daly, 2005). More details on the implementation can be found in section 4. A file with all the algorithms can be downloaded from http://www.dcc.ufrj.br/~collier/Folia.html.

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