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Effective bounds for the consistency of differential equations

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ABSTRACT

One method to determine whether or not a system of partial differential equations is consistent is to attempt to construct a solution using merely the "algebraic data" associated to the system. In technical terms, this translates to the problem of determining the existence of regular realizations of differential kernels via their possible prolongations. In this paper we effectively compute an improved upper bound for the number of prolongations needed to guarantee the existence of such realizations, which ultimately produces solutions to many types of systems of partial differential equations. This bound has several applications, including an improved upper bound for the order of characteristic sets of prime differential ideals. We obtain our upper bound by proving a new result on the growth of the Hilbert–Samuel function, which may be of independent interest.

1. Introduction

In this paper we study techniques that effectively determine if a given system of algebraic partial differential equations is consistent; that is, if the system has a solution in a differential field extension of the ground differential field in which the coefficients of the system live. Our approach is to study the set of algebraic solutions of a given system of algebraic differential equations (viewed as a purely

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algebraic system), and then determine if an algebraic solution can be used to construct a differential solution. This construction is not always possible, as evidenced by very basic examples such as the following:

$$\begin{cases} \partial_1 u = u \\ \partial_2 u = 1 \end{cases} \tag{1.1}$$

where u is a differential indeterminate over some ground differential field with two commuting derivations ∂_1 and ∂_2. If we consider the associated algebraic system obtained by replacing u, $\partial_1 u$, and $\partial_2 u$ with algebraic indeterminates x, z_1, and z_2, respectively, we obtain

$$\begin{cases} z_1 = x \\ z_2 = 1, \end{cases}$$

which has a solution. However, the differential system (1.1) is inconsistent, since the existence of a differential solution a in some differential field would imply $1 = \partial_2\partial_1 a = \partial_1\partial_2 a = 0$. It is important to note that the inconsistency of the system becomes apparent after differentiating the system once. The number of differentiations needed to reveal that a given system is inconsistent is the main motivation of this paper. Furthermore, we seek to effectively determine this number from data obtained from the equations (their order and the number of derivations and indeterminates).

To make the above discussion more precise, we study *differential kernels*, which are field extensions of the ground differential field $(K, \partial_1, \dots, \partial_m)$ obtained by adjoining a solution of the associated algebraic system such that this solution serves as a means to "prolong" the derivations from K (see Definition 1 for the precise definition of differential kernels). Differential kernels in a single derivation were studied by Cohn (1979) and Lando (1970). In Section 2, we consider differential kernels with an arbitrary number of commuting derivations. A differential kernel is said to have a *regular realization* if there is a differential field extension of K containing the differential kernel and such that the generators of the kernel form the sequence of derivatives of the generators of order zero. The key observation is that a differential kernel has a regular realization if and only if the chosen solution of the associated algebraic system (i.e., the generators of the differential kernel) can be prolonged to yield a differential solution to the original system of differential equations. Thus, the problem of determining the consistency of a given system of differential equations is equivalent to the problem of determining the existence of regular realizations of a given differential kernel. In a single derivation, every differential kernel has a regular realization (Lando, 1970, Proposition 3). However, this is no longer the case with more than one derivation, as evidenced by the system (1.1) above, which is also discussed in Example 3 below.

The first analysis of differential kernels with several commuting derivations appears in the work of Pierce (2014), using different terminology (there a differential kernel is referred to as a field extension satisfying the *differential condition*). In that paper it is shown that if a differential kernel has a *prolongation* of a certain length (that is, we can extend the derivations from the algebraic solution some finite number of times), then it has a regular realization; see Theorem 11 below. We note here that even if a differential kernel has a proper prolongation, this is no guarantee that a regular realization will exist, as evidenced by Example 9 below. We denote by $T^n_{r,m}$ the smallest prolongation length that guarantees the existence of a regular realization of any differential kernel of length r in n differential indeterminates over any differential field of characteristic zero with m commuting derivations; see Definition 12. Note that this number only depends on the data (r, m, n); in particular, it does *not* depend on the degree of the algebraic system associated to the differential kernel. A recursive construction of an upper bound for $T^n_{r,m}$ was provided in León Sánchez and Ovchinnikov (2016, §3); unfortunately, this upper bound is unwieldy from a computational standpoint even when $m = 2$ or 3.

In this paper, we provide a new and improved upper bound for $T^n_{r,m}$. This new upper bound is given in Theorem 18 by the number $C^n_{r,m}$, which we introduce in Section 3. The central idea for the construction of $C^n_{r,m}$ comes from weakening a condition imposed on what are called the minimal leaders of a differential kernel that guarantees the existence of a regular realization (compare conditions (†) and (♯)). In further sections we show that there is a recursive algorithm that computes the

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