ARTICLE IN PRESS YJSCO:1819

Journal of Symbolic Computation ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Flat extension and ideal projection ☆

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ARTICLE INFO

Article history:
Received 9 August 2017
Accepted 2 October 2017
Available online xxxx

Keywords:
Flat extension
Ideal
Projection
Multivariate
Moment problem
Moment matrix
Hankel operator

ABSTRACT

A generalization of the flat extension theorems of Curto and Fialkow and Laurent and Mourrain is obtained by seeing the problem as one of ideal projection.

1. Introduction

The Hamburger moment problem is to determine the existence and uniqueness of a positive measure whose moments

$$\int_{\mathbb{R}^k} x^\alpha \, d\mu, \quad \alpha \in \mathbb{Z}_+^k,$$

take a prescribed multi-sequence of values $(y_\alpha)_{\alpha\in\mathbb{Z}_+^k}$. Here, x^α is the αth monomial in Π, the vector space of all polynomials on $\mathbb{R}^k$. For such a measure to exist, it is necessary and sufficient that the linear functional

$$L : \Pi \to \mathbb{R} : p = \sum_\alpha \hat{p}(\alpha)x^\alpha \mapsto \sum_\alpha \hat{p}(\alpha)y_\alpha$$

☆ This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.
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https://doi.org/10.1016/j.jsc.2017.11.007
0747-7171/

be nonnegative, that is, $Lp \geq 0$ if $p \geq 0$, and for this to occur, it is necessary that the **moment matrix**

$$[y_{\alpha+\beta}]_{\alpha,\beta\in\mathbb{Z}_+^k} \tag{1}$$

be positive semidefinite, that is, $L(p^2) \geq 0$ for all p. This condition is sufficient only under special circumstances, for instance, when every positive polynomial can be written as a sum of squares, as occurs when $k = 1$ (Pólya and Szego, 1976, §6.6) and again when (1) has finite rank, since in that case the functional L is finitely atomic, i.e., a linear combination of finitely many shifts of δ and its derivatives (Laurent and Rostalski, 2012, Theorem 7).

Finitely atomic measures and their moment matrices are the subject of the **truncated moment problem** of Curto and Fialkow (1996), who address when such a measure is determined by finitely many of its moments. To be precise, Curto and Fialkow work in $\mathbb{C}^k$ and study the relationship between the complex moment matrix

$$\left[\gamma_{\alpha,\beta} := \int_{\mathbb{C}^k} \bar{x}^\alpha x^\beta \, d\mu\right]_{\alpha,\beta\in\mathbb{Z}_+^k}, \tag{2}$$

its submatrices of the form

$$\left[\gamma_{\alpha,\beta}\right]_{|\alpha|,|\beta|\leq n}, \tag{3}$$

and the size of the support of μ. One of their main results is that if the positive semidefinite (3) can be extended **flatly**, meaning without increasing its rank, to a positive semidefinite

$$\left[\gamma_{\alpha,\beta}\right]_{|\alpha|,|\beta|\leq n+1},$$

then the later has a unique, positive semidefinite, flat extension (2) and a finitely-atomic representing measure μ, and the rank of the matrix equals the cardinality of the measure's support.

Curto and Fialkow's flat extension theorem in the special case when $\mathbb{C}$ is replaced by $\mathbb{R}$ is generalized by Laurent and Mourrain (2009). Working with moment matrices of the form (1), they replace $\mathbb{C}$ by any field $\mathbb{F}$ and do not require moment matrices to be positive semidefinite. They then prove that if the finite set $C \subset \mathbb{Z}_+^k$ is **connected to one**, meaning

$$C \neq \emptyset, \text{ and } \alpha \in C \setminus 0 \Longrightarrow \alpha - e_j \in C \text{ for some } j, \tag{4}$$

and if

$$[y_{\alpha+\beta}]_{\alpha,\beta\in C} \tag{5}$$

can be flatly extended to

$$[y_{\alpha+\beta}]_{\alpha,\beta\in C^+} \tag{6}$$

where

$$C^+ := C \cup \{\alpha + e_j : \alpha \in C, j = 1:k\}$$

and $e_j \in \mathbb{F}^k$ is the vector given by

$$(e_j)_i = \delta_{i,j}, \quad i, j = 1:k, \tag{7}$$

then (6) has a unique flat extension (1). In case $\mathbb{F}$ is $\mathbb{R}$, if (6) is positive semidefinite, then so is (1).

(To be consistent with Laurent and Mourrain, we'll say that the monomial space $\Pi_C := \text{span}\{x^\alpha : \alpha \in C\}$ is connected to one if the set $C \subset \mathbb{Z}_+^k$ is connected to one.)