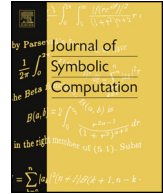




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Elliptic function based algorithms to prove Jacobi theta function relations[☆]

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ABSTRACT

In this paper we prove identities involving the classical Jacobi theta functions of the form

$$\sum c(i_1, i_2, i_3, i_4) \theta_1(z|\tau)^{i_1} \theta_2(z|\tau)^{i_2} \theta_3(z|\tau)^{i_3} \theta_4(z|\tau)^{i_4} = 0$$

with $c(i_1, i_2, i_3, i_4) \in \mathbb{K}[\Theta]$, where $\mathbb{K}$ is a computable field and $\Theta := \{\theta_1^{(2k+1)}(0|\tau) : k \in \mathbb{N}\} \cup \{\theta_j^{(2k)}(0|\tau) : k \in \mathbb{N} \text{ and } j = 2, 3, 4\}$. We give two algorithms that solve this problem. The second algorithm is simpler and works in a restricted input class.

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1. Introduction

Our ultimate goal is to develop computer-assisted treatment for identities among Jacobi theta functions, namely, to automatize the proving procedures of relations and the discovery of relations.

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Let us recall the definition of Jacobi theta functions $\theta_j(z|\tau)$ ($j = 1, \dots, 4$):

Definition 1.1. (DLMF, 2015, 20.2(i)) Let $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ and $q := e^{\pi i \tau}$, then

$$\theta_1(z, q) := \theta_1(z|\tau) := 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin((2n+1)z),$$

$$\theta_2(z, q) := \theta_2(z|\tau) := 2 \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} \cos((2n+1)z),$$

$$\theta_3(z, q) := \theta_3(z|\tau) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz),$$

$$\theta_4(z, q) := \theta_4(z|\tau) := 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz).$$

As a first step towards the goal we mentioned in the beginning, in Ye (2017) we provided an algorithm to prove identities involving the derivatives of $\theta_j(z|\tau)$ ($j = 1, 2, 3, 4$), in particular, involving

$$\theta_j^{(k)} := \theta_j^{(k)}(0|\tau) := \left. \frac{\partial^k \theta_j}{\partial z^k}(z|\tau) \right|_{z=0}, \quad k \in \mathbb{N} := \{0, 1, 2, \dots\}.$$

For example, Algorithm 5.11 of Ye (2017) can assist us to prove identities like

$$\theta_3^{(4)} \theta_3 - 3(\theta_3'')^2 - 2\theta_3^2 \theta_2^4 \theta_4^4 = 0$$

from Rademacher (1973, (93.22)),

$$\frac{\theta_\alpha^{(5)}}{\theta_1'} - 3 \left(\frac{\theta_\alpha''}{\theta_\alpha} \right)^2 + 2 \left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\beta''}{\theta_\beta} \right) \left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\gamma''}{\theta_\gamma} \right) = 0$$

from Rademacher (1973, (93.7)), where $\alpha, \beta, \gamma = 2, 3, 4$, and

$$\frac{\theta_1^{(3)}}{\theta_1'} - \frac{\theta_2''}{\theta_2} - \frac{\theta_3''}{\theta_3} - \frac{\theta_4''}{\theta_4} = 0$$

from Lawden (1989, p. 22).

More generally, in Ye (2017) we showed that this algorithm can do zero-recognition on any function in $\mathbb{K}[\Theta]$, which is the $\mathbb{K}$ -algebra generated by

$$\Theta := \left\{ \theta_1^{(2k+1)}(0|\tau) : k \in \mathbb{N} \right\} \cup \left\{ \theta_j^{(2k)}(0|\tau) : k \in \mathbb{N} \text{ and } j = 2, 3, 4 \right\},$$

where $\mathbb{K} \subseteq \mathbb{C}$ is an effectively computable field which contains all the complex constants we need (i.e., i , $e^{\pi i/4}$, etc.). The reason why we omit $\theta_1^{(k_1)}(0|\tau)$ when $k_1 \in 2\mathbb{N}$, and omit $\theta_m^{(k_2)}(0|\tau)$ ($m = 2, 3, 4$) when $k_2 \in 2\mathbb{N} + 1$ is that by Definition 1.1 these are equal to zero.

In this article we extend the function space $\mathbb{K}[\Theta]$ to

$$R_1 := \mathbb{K}[\Theta][\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau), \theta_4(z|\tau)],$$

which is the $\mathbb{K}[\Theta]$ -algebra generated by $\theta_1(z|\tau)$, $\theta_2(z|\tau)$, $\theta_3(z|\tau)$ and $\theta_4(z|\tau)$. In particular, we solve the following problem algorithmically:

Problem 1.1. Given $f \in R_1$, decide whether $f = 0$.

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