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Elliptic function based algorithms to prove Jacobi theta function relations ☆

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ABSTRACT

In this paper we prove identities involving the classical Jacobi theta functions of the form

$$\sum c(i_1, i_2, i_3, i_4)\theta_1(z|\tau)^{i_1}\theta_2(z|\tau)^{i_2}\theta_3(z|\tau)^{i_3}\theta_4(z|\tau)^{i_4} = 0$$

with $c(i_1, i_2, i_3, i_4) \in \mathbb{K}[\Theta]$, where $\mathbb{K}$ is a computable field and $\Theta := \left\{\theta_1^{(2k+1)}(0|\tau) : k \in \mathbb{N}\right\} \cup \left\{\theta_j^{(2k)}(0|\tau) : k \in \mathbb{N} \text{ and } j = 2, 3, 4\right\}$. We give two algorithms that solve this problem. The second algorithm is simpler and works in a restricted input class.

1. Introduction

Our ultimate goal is to develop computer-assisted treatment for identities among Jacobi theta functions, namely, to automatize the proving procedures of relations and the discovery of relations.

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Let us recall the definition of Jacobi theta functions $\theta_j(z|\tau)$ $(j=1,\dots,4)$:

Definition 1.1. (DLMF, 2015, 20.2(i)) Let $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $q := e^{\pi i \tau}$, then

$$\theta_1(z,q) := \theta_1(z|\tau) := 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin((2n+1)z),$$

$$\theta_2(z,q) := \theta_2(z|\tau) := 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos((2n+1)z),$$

$$\theta_3(z,q) := \theta_3(z|\tau) := 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz),$$

$$\theta_4(z,q) := \theta_4(z|\tau) := 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz).$$

As a first step towards the goal we mentioned in the beginning, in Ye (2017) we provided an algorithm to prove identities involving the derivatives of $\theta_j(z|\tau)$ $(j=1,2,3,4)$, in particular, involving

$$\theta_j^{(k)} := \theta_j^{(k)}(0|\tau) := \frac{\partial^k \theta_j}{\partial z^k}(z|\tau)\bigg|_{z=0}, \quad k \in \mathbb{N} := \{0,1,2,\dots\}.$$

For example, Algorithm 5.11 of Ye (2017) can assist us to prove identities like

$$\theta_3^{(4)}\theta_3 - 3(\theta_3'')^2 - 2\theta_3^2\theta_2^4\theta_4^4 = 0$$

from Rademacher (1973, (93.22)),

$$\frac{\theta_\alpha^{(5)}}{\theta_1'} - 3\left(\frac{\theta_\alpha''}{\theta_\alpha}\right)^2 + 2\left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\beta''}{\theta_\beta}\right)\left(\frac{\theta_\alpha''}{\theta_\alpha} - \frac{\theta_\gamma''}{\theta_\gamma}\right) = 0$$

from Rademacher (1973, (93.7)), where $\alpha, \beta, \gamma = 2,3,4$, and

$$\frac{\theta_1^{(3)}}{\theta_1'} - \frac{\theta_2''}{\theta_2} - \frac{\theta_3''}{\theta_3} - \frac{\theta_4''}{\theta_4} = 0$$

from Lawden (1989, p. 22).

More generally, in Ye (2017) we showed that this algorithm can do zero-recognition on any function in $\mathbb{K}[\Theta]$, which is the $\mathbb{K}$-algebra generated by

$$\Theta := \left\{\theta_1^{(2k+1)}(0|\tau) : k \in \mathbb{N}\right\} \cup \left\{\theta_j^{(2k)}(0|\tau) : k \in \mathbb{N} \text{ and } j = 2,3,4\right\},$$

where $\mathbb{K} \subseteq \mathbb{C}$ is an effectively computable field which contains all the complex constants we need (i.e., i, $e^{\pi i/4}$, etc.). The reason why we omit $\theta_1^{(k_1)}(0|\tau)$ when $k_1 \in 2\mathbb{N}$, and omit $\theta_m^{(k_2)}(0|\tau)$ $(m=2,3,4)$ when $k_2 \in 2\mathbb{N}+1$ is that by Definition 1.1 these are equal to zero.

In this article we extend the function space $\mathbb{K}[\Theta]$ to

$$R_1 := \mathbb{K}[\Theta][\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau), \theta_4(z|\tau)],$$

which is the $\mathbb{K}[\Theta]$-algebra generated by $\theta_1(z|\tau), \theta_2(z|\tau), \theta_3(z|\tau)$ and $\theta_4(z|\tau)$. In particular, we solve the following problem algorithmically:

Problem 1.1. Given $f \in R_1$, decide whether $f = 0$.