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Combinatorial rigidity of incidence systems and application to dictionary learning ☆

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A B S T R A C T

Given a hypergraph H with m hyperedges and a set Q of m *pinning subspaces*, i.e. globally fixed subspaces in Euclidean space $\mathbb{R}^d$, a *pinned subspace-incidence system* is the pair (H, Q), with the constraint that each pinning subspace in Q is contained in the subspace spanned by the point realizations in $\mathbb{R}^d$ of vertices of the corresponding hyperedge of H. For a subclass of pinned subspace-incidence systems where all pinning subspaces are of dimension 1, this paper provides a combinatorial characterization of *minimal rigidity*, i.e. those systems that are guaranteed to generically yield a locally unique realization.

Pinned subspace-incidence systems arise as a geometric interpretation of the *Dictionary Learning (aka sparse coding)* problem, i.e. the problem of obtaining a sparse representation of a given set of data vectors by learning *dictionary vectors* upon which the data vectors can be written as sparse linear combinations. Viewing the dictionary vectors from a geometry perspective as the spanning set of a subspace arrangement, we provide a systematic classification of problems related to dictionary learning together with various algorithms, their assumptions and performance. We formally prove the intuitively expected bound that the size of dictionary cannot be significantly less than the number of data vectors when the data are generic or uniformly distributed, and gives a way of constructing a dictionary that meets the bound. For less stringent restrictions on data, but a natural modification of the dictionary learning problem, we provide a further dictionary learning algorithm by leveraging the well-known *DR-planning* technique from geometric constraint solving. Although there are recent rigidity based approaches for low rank matrix completion, we are unaware of prior

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application of combinatorial rigidity techniques in the setting of Dictionary Learning.
Other applications of pinned subspace-incidence systems include modeling microfibrils in biomaterials such as cellulose and collagen.

1. Introduction

A *pinned subspace-incidence system* (H, Q) is an incidence constraint system specified as a hypergraph H together with a set Q of *pinning subspaces* in $\mathbb{R}^d$, each specified by a collection of basis vectors called *pins*. The pinning subspaces are in one-to-one correspondence with the hyperedges of H. A realization of (H, Q) is a subspace arrangement that assigns vectors in $\mathbb{R}^d$ to the vertices of H. Each hyperedge of H is assigned the subspace spanned by its vertex vectors and contains the associated pinning subspace in Q. In this paper, we study the rigidity of pinned subspace-incidence systems. In particular, we provide combinatorial rigidity characterization for a subclass of pinned subspace-incidence systems where any set of $d-1$ or fewer vertices contain at most one pin.

We are unaware of any previous combinatorial rigidity results on pinned-subspace incidence systems. Previous work on related types of frameworks include pin-collinear body-pin frameworks (Jackson and Jordán, 2008), direction networks (Whiteley, 1996), slider-pinning rigidity (Streinu and Theran, 2010), body-cad constraint system (Haller et al., 2012), k-frames (White and Whiteley, 1987, 1983), and affine rigidity (Gortler et al., 2013). All of which involve some form of incidence constraints.

1.1. Connection to solving polynomial systems and automated deduction for incidence constraints

Finding all solutions of polynomial systems arising from geometric constraints is an intractable problem unless the systems are recursively decomposed into subsystems with finitely many isolated solutions that are then recombined recursively. An optimal recursive decomposition minimizes the size of the algebraic system that needs to be simultaneously solved. The recursive decomposition is best done by employing combinatorial rigidity based algorithms and is called *DR-planning* (Hoffman et al., 2001b). Even then, finding an optimal decomposition of the system is typically NP-hard. In (Baker et al., 2015) it has been shown that under certain conditions, the problem can be solved in $O(n^3)$ time, technically when the underlying abstract rigidity matroid happens to be a sparsity matroid – a property that pinned subspace incidence systems have – and when the system is generic and independent. However, there is no practical alternative to DR-planning even when the system is dependent or overconstrained, and the (Baker et al., 2015) algorithm still works except it does not guarantee efficiency in finding the optimal DR-plan.

The main obstacle in determining rigidity of pinned subspace incidence systems is determining (in)dependence of the incidence constraints represented by the pinning subspaces. A dependent constraint is essentially an incidence theorem. In that sense, determining rigidity in pinned-subspace incidence systems is related to incidence geometry theorem proving (Li and Wu, 2001), where Grassman–Cayley bracket algebra is used for general, nongeneric settings. Here, we restrict ourselves to *generic* incidence theorems that hold for generic frameworks and that can be captured using the combinatorics of the incidence hypergraph, using established techniques in combinatorial rigidity.

1.2. Connection to dictionary learning

Pinned subspace-incidence systems naturally arise as a geometric interpretation of *dictionary learning (aka sparse coding)* problem (Olshausen and Field, 1997), i.e. the problem of obtaining a sparse