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The hypermetric cone and polytope on eight vertices and some generalizations ☆

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ABSTRACT

The paper deals with geometric constraints on Delaunay polytopes, arising from hypermetric inequalities with origins in lattice theory. In some cases the constraints are sufficient to uniquely define a Delaunay polytope, a situation of primary interest in combinatorial rigidity; and the configuration space of underconstrained Delaunay polytopes defines a face of the hypermetric cone. Symbolic algorithms and computations algorithms form the basis of the paper's results and illustrative examples.

The lists of facets – 298, 592 in 86 orbits – and of extreme rays – 242, 695, 427 in 9, 003 orbits – of the hypermetric cone HYP_8 are computed. The notion of hypermetric occurs in Metric Geometry and realization spaces of Delaunay polytopes in lattices and we consider a number of generalizations.

The first one is the hypermetric polytope $HYPP_n$, for which we give general algorithms and a description for $n \leq 8$. We give a complete theory of it and of its link to centrally symmetric Delaunay polytope.

Then we shortly consider generalizations to the case of lattice Delaunay simplices of index higher than 1. The case of hypermetrics on graphs is also considered and we show how one can obtain new valid inequalities for the cut-polytope of a graph. We then consider shortly the case of infinite hypermetrics.

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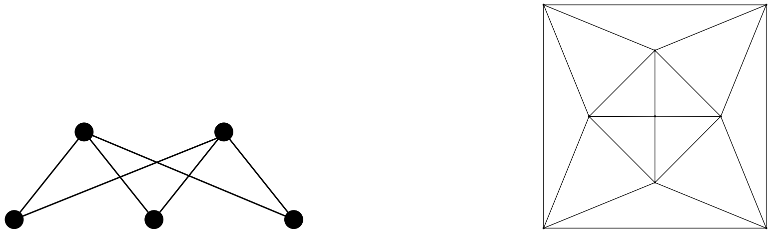

A graph G with $d_G \in MET_5$ and $d_G \notin HYP_5$.

A graph G with $d_G \in HYP_9$ and $d_G \notin CUT_9$.

Fig. 1. Two interesting graphs and properties of their shortest path distances d_G.

1. Introduction

Metric, cut and hypermetric cones are among central objects of Discrete Mathematics. For example, finite metrics and l_1-metrics can be studied by polyhedral cones and polytopes; see, say, Deza and Laurent (2010).

The *hypermetric cone* HYP_n is the set of all *hypermetrics on n points*, i.e., the functions $d : \{1, \dots, n\}^2 \to \mathbb{R}$, such that $d(i,i) = 0$, $d(i,j) = d(j,i)$ and

$$H(b,d) = \sum_{1 \le i < j \le n} b_i b_j d(i,j) \le 0 \text{ for all } b \in \mathbb{Z}^n, \sum_{i=1}^{n} b_i = 1.$$

The *metric cone* MET_n is the set of all *semimetrics on n points*, i.e., above functions d, which satisfy only all *triangle inequalities*:

$$d(i,j) \le d(i,k) + d(k,j) \text{ for } 1 \le i, j, k \le n.$$

Note that triangle inequalities are hypermetric inequalities for $b = (1, 1, -1, 0^{n-3})$ and its permutations.

For a set $S \subseteq \{1, \dots, n\}$, the *cut* (or *split*) *semimetric* δ_S is a vector defined as

$$\delta_S(x,y) = \begin{cases} 1 & \text{if } |S \cap \{x,y\}| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\delta_{\overline{S}} = \delta_S$, and δ_S can be seen also as the adjacency matrix of a *cut* (into S and $\overline{S}$) *subgraph* of K_n. The *cut cone* CUT_n is the positive span of the $2^{n-1} - 1$ non-zero cut semimetrics; the *cut polytope* $CUTP_n$ is the convex hull of all 2^{n-1} cut semimetrics. We have the inclusions $CUT_n \subseteq HYP_n \subseteq MET_n$, as well as the inclusions $CUTP_n \subseteq HYPP_n \subseteq METP_n$ for the polytopes $METP_n$ and $HYPP_n$ defined later. In Fig. 1 we give examples of distances showing that the inclusions are strict. We have $CUT_n = MET_n$ only for $3 \le n \le 4$; also, $CUT_n = HYP_n$ only for $3 \le n \le 6$ (Deza and Laurent, 2010); see Table 1 for the number of facets and extreme rays of CUT_n, HYP_n and MET_n for $n \le 8$.

The cut polytope is directly related to the Max-Cut Problem (Korte and Vygen, 2008) which is a classic problem of Combinatorial Optimization on graphs. The Max-Cut Problem is used for l_1-embedding questions (Deza et al., 2004) of graphs, as well as for application in Analysis, Combinatorics, Combinatorial Optimization (Deza and Laurent, 2010), and Statistical Mechanics (Anglès d'Auriac et al., 1997). A finite metric is hypermetric if and only if it corresponds to the squared Euclidean distance between vertices of a Delaunay polytope of index 1; see Section 3 for details. In Section 5 we generalize this result and establish the link between the hypermetric polytope $HYPP_n$ and n-dimensional centrally symmetric Delaunay polytopes. In Section 6, we consider the generalization of hypermetric cone to Delaunay polytope of index higher than 1.