ARTICLE IN PRESS YJSCO:1787

Journal of Symbolic Computation ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Difference indices of quasi-prime difference algebraic systems

Jie Wang

KLMM, Academy of Mathematics and Systems Science, The Chinese Academy of Sciences, Beijing 100190, China

ARTICLE INFO

Article history:
Received 24 July 2016
Accepted 18 May 2017
Available online xxxx

MSC:
primary 12H10
secondary 68W30

Keywords:
Difference index
Order of difference ideal
Hilbert–Levin regularity
Difference ideal membership problem

ABSTRACT

This paper is devoted to studying difference indices of quasi-prime difference algebraic systems. We define the quasi dimension polynomial of a quasi-prime difference algebraic system. Based on this, we give the definition of the difference index of a quasi-prime difference algebraic system through a family of pseudo-Jacobian matrices. Some properties of difference indices are proved. In particular, an upper bound for difference indices is given. As applications, an upper bound for the Hilbert–Levin regularity and an effective difference ideal membership theorem for quasi-prime difference algebraic systems are deduced.

1. Introduction

The main notion we consider in this paper is the *difference index* of a difference algebraic system over a difference field K (i.e. a field with a transforming operator, for instance $K := \mathbb{Q}(x)$ the field of rational functions with the shift $\sigma : f(x) \mapsto f(x+1)$ as a transforming operator). Roughly speaking, the difference index is an important numerical invariant associated to a difference algebraic system which provides the order of transform we need to apply to the system to obtain the relations up to a prescribed order that all the solutions must verify. In some sense, difference indices can be regarded as a measure of the complexity of difference algebraic systems. The difference index is also closely related to some other important invariants of a difference algebraic system, for example, the order and the Hilbert–Levin regularity. Moreover, difference indices can be utilized to solve the difference ideal membership problem.

E-mail address: wangjie212@mails.ucas.ac.cn.

http://dx.doi.org/10.1016/j.jsc.2017.05.004

The analogous notion for a differential algebraic system is the differential index, which has been extensively studied for many years. Actually, there are several definitions of differential indices of a differential algebraic system in the literature (see for instance D'Alfonso et al., 2009, 2008; Campbell and Gear, 1995; Le Vey, 1994; Pantelides, 1988; Seiler, 1999). Although they are not completely equivalent, in each case they represent a measure of the implicitness of the given system. However, the corresponding notion of difference indices for difference algebraic systems has been rarely studied yet. Recently in Wang (2016), difference indices of quasi-regular difference algebraic systems were first defined following the analogous method used in D'Alfonso et al. (2009, 2008) by D'Alfonso, Jeronimo, Massaccesi and Solernó. In this paper, we will generalize the definition of difference indices to more general difference algebraic systems, i.e. quasi-prime difference algebraic systems. The difficulty is to calculate the transcendence degrees of certain associated field extensions without the regular condition. In order to overcome this difficulty, we will introduce the new concept of quasi dimension polynomials for quasi-prime difference algebraic systems. Let us explain it in more details.

Suppose F is a set of difference polynomials, Δ is the difference ideal generated by F, and $\mathfrak{p}$ is a minimal reflexive prime difference ideal over Δ. Denote by Δ_k the algebraic ideal generated by F and the transforms of F up to the order $k-1$ in the corresponding localized polynomial ring at $\mathfrak{p}$. Then we say the system F is *quasi-prime* at $\mathfrak{p}$ if Δ_k is a prime ideal for any positive integer k and Δ is reflexive. For a difference algebraic system F which is quasi-prime at $\mathfrak{p}$, we consider the dimension of Δ_k as a function of k, denoted by $\psi(k)$. It turns out that $\psi(k)$ becomes a polynomial of degree one for k large enough, which we call the $\mathfrak{p}$-*quasi dimension polynomial* of the system F. By virtue of $\mathfrak{p}$-quasi dimension polynomials, we can give the definition of the difference index of a quasi-prime difference algebraic system, which is called the $\mathfrak{p}$-*difference index*. As usual, its definition follows from a certain chain which eventually becomes stationary. In analogy with the case of $\mathfrak{P}$-differential indices in D'Alfonso et al. (2009) and the case of $\mathfrak{p}$-difference indices in Wang (2016), the chain is established by the sequence of ranks of certain Jacobian submatrices associated with the system F. Assume F is quasi-prime at Δ, ω is the Δ-difference index of the system F and ρ is the least k such that the Δ-quasi dimension polynomial of F holds. Then it turns out that for $i+\omega \geq \rho+e-1$ (e is the highest order of F), ω satisfies:

$$\Delta_{i-e+1+\omega} \cap A_i = \Delta \cap A_i,$$

where A_i is the polynomial ring in the variables with orders no more than i, which meets our expectation for difference indices.

This approach enables us to give an upper bound for the $\mathfrak{p}$-difference index of a quasi-prime difference algebraic system. Basing on this, we can give several applications of $\mathfrak{p}$-difference indices, including an upper bound for the Hilbert–Levin regularity and an upper bound of orders for the difference ideal membership problem of a quasi-prime difference algebraic system.

The paper will be organized as follows. In Section 2, we list some basic notions from difference algebra which will be used later. In Section 3, the $\mathfrak{p}$-quasi dimension polynomial of a quasi-prime difference algebraic system is defined. In Section 4, we introduce a family of pseudo-Jacobian matrices and give the definition of $\mathfrak{p}$-difference indices through studying the ranks of them. In Section 5, some properties of $\mathfrak{p}$-difference indices will be proved. In Section 6, several applications of $\mathfrak{p}$-difference indices are given. In Section 7, we give an example.

2. Preliminaries

A *difference ring* or σ-*ring* for short (R, σ), is a commutative ring R together with a ring endomorphism $\sigma: R \to R$. If R is a field, then we call it a *difference field*, or a σ-*field* for short. We call σ the *transforming operator* of R and usually omit σ from the notation, simply refer to R as a σ-ring or a σ-field. A typical example of σ-field is the field of rational functions $\mathbb{Q}(x)$ with $\sigma(f(x)) = f(x+1)$. For any $a \in R$, $\sigma(a)$ is called the *transform* of a. For $n \in \mathbb{N}$, $\sigma^n(a) = \sigma^{n-1}(\sigma(a))$ is called the *n-th transform* of a, with the usual assumption $\sigma^0(a) = a$. In this paper, unless otherwise specified, K is always assumed to be a σ-field of characteristic 0.

Please cite this article in press as: Wang, J. Difference indices of quasi-prime difference algebraic systems. J. Symb. Comput. (2017), http://dx.doi.org/10.1016/j.jsc.2017.05.004