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Sparse resultants and straight-line programs ☆

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We prove that the sparse resultant, redefined by D'Andrea and Sombra and by Esterov as a power of the classical sparse resultant, can be evaluated in a number of steps which is polynomial in its degree, its number of variables and the size of the exponents of the monomials in the Laurent polynomials involved in its definition. Moreover, we design a probabilistic algorithm of this order of complexity to compute a straight-line program that evaluates it within this number of steps.

1. Introduction

Resultants are considered a key tool in the resolution of polynomial equation systems, mainly because of their role as eliminating polynomials. In the last decades, the practical utility of resultants has aroused interest in their effective computation.

The study of classical homogeneous resultants goes back to Bézout, Cayley and Sylvester (see Bézout, 1779, Cayley, 1848 and Sylvester, 1853). In Macaulay (1902), Macaulay obtained explicit formulae for the homogeneous resultant as a quotient of two determinants and, from then on, several effective procedures to compute these resultants have been proposed (see, for example, D'Andrea and Dickenstein, 2001 and the references therein). More recently, Gelfand, Kapranov and Zelevinski gen-

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eralized the classical notion to the sparse setting (see Gelfand et al., 1994). The first effective method for computing sparse resultants was given in Sturmfels (1993). In Canny and Emiris (1993) (see also Canny and Emiris, 2000) and Sturmfels (1994), the authors provided an algorithm for computing a square Sylvester style matrix with determinant equal to a nonzero multiple of the resultant. A survey of matrix constructions for the computation of resultants can be found in Emiris and Mourrain (1999). In D'Andrea (2002), it was shown that the sparse resultant is a quotient of the determinant of a Sylvester style matrix by one of its minors, extending Macaulay's formulation to the sparse setting.

When dealing with the computation of the resultant of a particular system in order to know whether it vanishes or not, its representation as a quotient of determinants may not be enough because the denominator may vanish. Classical methods to solve this problem consist in making a symbolic perturbation to the system (see, for instance, Canny and Emiris, 1993), but they require further computations for each particular system. This motivates the search for a division-free representation of the resultant. A possible approach to do this is the classical Strassen's method to eliminate divisions described in Strassen (1973). Using this method, in Kaltofen and Koiran (2008), it is shown how to express a quotient of two determinants that is a polynomial as a single determinant of a matrix of size polynomial in the sizes of the original matrices.

All the previously mentioned procedures for the computation of sparse resultants deal with matrices of exponential size: for $n+1$ Laurent polynomials in n variables with n-dimensional Newton polytopes, the number of rows and columns of the matrices involved in the computation of the associated resultant is of order $O(k^n n^{-3/2} D)$ (see Canny and Emiris, 2000, Theorem 3.10), where k is a positive constant and D is the total degree of the resultant as a polynomial in the coefficients of the input. This implies that the algebraic complexity of any algorithm using these matrices is necessarily exponential in the number of variables n of the input polynomials. For the classical homogeneous resultant, the complexity of testing it for zero and of the algorithms to compute it via Macaulay matrices was studied in Grenet et al. (2013).

Due to the well-known estimates for the degree of the sparse resultant in terms of mixed volumes (see, for instance, Pedersen and Sturmfels, 1993, Corollary 2.4, and D'Andrea and Sombra, 2015, Proposition 3.4), any algorithm for its computation which encodes it as an array of coefficients (dense form) cannot have a polynomial complexity in the size of the input (that is, the number of coefficients of the generic polynomial system whose resultant is computed). Then, in order to obtain this polynomial order of complexity, a different way of representing polynomials should be used.

An alternative data structure which was introduced in the polynomial equation solving framework yielding a significant reduction in the previously known complexities is the *straight-line program* representation of polynomials (see, for instance, Giusti and Heintz, 1993 and Giusti et al., 1998, where this data structure allowed the design of the first algorithms for solving zero-dimensional polynomial systems within complexity polynomial in the output size). Roughly speaking, a straight-line program which encodes a polynomial is a program which enables us to evaluate it at any given point. The first algorithm for the computation of (homogeneous and) sparse resultants using straight-line programs was presented in Jeronimo et al. (2004). Its complexity is polynomial in the dimension of the ambient space and the volume associated to the input set of exponents, but it deals only with a subclass of unmixed resultants. Afterwards, in Jeronimo and Sabia (2007), an algorithm for the computation of both mixed and unmixed multihomogeneous resultants by means of straight-line programs was given. The algorithm relies on Poisson's product formula for the multihomogeneous resultant and its complexity is polynomial in the degree and the number of variables of the computed resultant.

The definition of the general sparse resultant for Laurent polynomials as an irreducible polynomial defining the corresponding incidence variety also implied a Poisson-type formula proved in Pedersen and Sturmfels (1993), but this formula does not hold for arbitrary supports. A restatement of this formula, valid in a more general setting, was given in Minimair (2003). In D'Andrea and Sombra (2015) (see also Esterov, 2010, Definition 3.1), the notion of sparse resultant was redefined and studied using multiprojective elimination theory. The new sparse resultant is a power of the previous one. It has better properties and produces more uniform statements, in particular, a nicer Poisson-type formula which holds for *any* family of supports (see D'Andrea and Sombra, 2015, Theorem 1.1). Further properties of this resultant have been studied in D'Andrea et al. (in preparation), where the Macaulay-style