ARTICLE IN PRESS YJSCO:1790

Journal of Symbolic Computation ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Computing linear systems on metric graphs

Bo Lin

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94805, USA

ARTICLE INFO

Article history:
Received 6 August 2016
Accepted 15 May 2017
Available online xxxx

Keywords:
Metric graph
Linear system
Anchor cell
f-vector

ABSTRACT

The linear system $|D|$ of a divisor D on a metric graph has the structure of a cell complex. We introduce the anchor divisors and anchor cells in it – they serve as the landmarks for us to compute the f-vector of the complex and find all cells in the complex. A linear system can also be identified as a tropical convex hull of rational functions. We compute its extremal generators using the landmarks. We apply these methods to some examples – namely the canonical linear systems of some small trivalent graphs.

1. Introduction

In algebraic geometry, the linear systems of divisors on curves are well studied (cf. Griffiths and Harris, 2014, §3). Other authors studied linear systems of divisors on metric graphs, for example (Baker, 2008; Baker and Faber, 2006; Haase et al., 2012). The article (Baker and Norine, 2007) proved a graph-theoretic analogue of the Riemann–Roch Theorem, and it was generalized to tropical curves (which may contain unbounded rays) independently by Gathmann and Kerber (2008) and by Mikhalkin and Zharkov (2008). The theory of linear systems on metric graphs is applied to algebraic geometry, for example in Baker et al. (2016).

Haase et al. (2012) studied the cell complex structure of $|D|$ and the tropical semi-module structure of $R(D)$, where D is a divisor on a metric graph. This work is an extension of Haase et al. (2012). We focus on the computation of the cell complex $|D|$, namely given a metric graph Γ and a divisor D on it, how to find the cells in $|D|$ and the f-vector of $|D|$. Since $|D|$ may contain a large number of cells and some of these are complicated, the complexity of computation could be high. We introduce the *anchor cell* that serves as the landmarks to find other cells in $|D|$. As a byproduct we can compute

E-mail address: linbo@math.berkeley.edu.

http://dx.doi.org/10.1016/j.jsc.2017.05.007

Please cite this article in press as: Lin, B. Computing linear systems on metric graphs. J. Symb. Comput. (2017), http://dx.doi.org/10.1016/j.jsc.2017.05.007

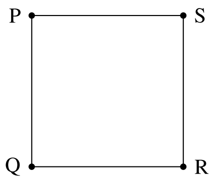

Fig. 1. The metric graph Γ.

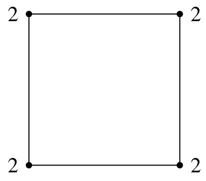

Fig. 2. The divisor D.

the extremal generators of $R(D)$. We implemented the algorithms and computed some examples – namely the canonical linear systems on some trivalent graphs.

A *metric graph* Γ is a connected undirected graph whose edges have positive lengths. A *divisor* D on Γ is a formal finite $\mathbb{Z}$-linear combination $D = \sum_{x\in\Gamma} D(x)\cdot x$ of points X in the edges of Γ. The divisor is *effective* if $D(x) \geq 0$ for all $x \in \Gamma$. The *degree* of a divisor D is $\sum_{x\in\Gamma} D(x)$. The *support* of a divisor D on Γ is the set $\{x \in \Gamma \mid D(x) \neq 0\}$, denoted as *supp*$(D)$.

A *(tropical) rational function* f on Γ is a continuous function $f : \Gamma \to \mathbb{R}$ that is piecewise-linear on each edge of Γ with finitely many pieces and integer slopes. The *order* $\mathrm{ord}_x(f)$ of f at a point $x \in \Gamma$ is the sum of the outgoing slopes at x along all directions. Note that if x is an interior point of a linear piece of f, then there are two directions at x, and x has two opposite outgoing slopes, so $\mathrm{ord}_x(f) = 0$. The *principal divisor* associated to f is

$$(f) = \sum_{x\in\Gamma} \mathrm{ord}_x(f)\cdot x.$$

So the support of (f) is always finite.

Two divisors D and D' are *linearly equivalent* if $D - D' = (f)$ for some rational function f, denoted as $D \sim D'$. For any divisor D on Γ, let $R(D)$ be the set of all rational functions f on Γ such that the divisor $D + (f)$ is effective, and $|D| = \{D + (f) \mid f \in R(D)\}$ be the *linear system* of D.

Example 1. Let Γ be a metric graph with graph-theoretic type C_4 and equal edge lengths. Figs. 1, 2, 3, 4 show Γ, D, a rational function of $f \in R(D)$ and the effective divisor $D + (f)$.

The metric graph Γ is determined by its graph-theoretic type and the lengths of its edges. The graph-theoretic type of Γ is called the *skeleton* of Γ, and the lengths of edges of Γ are called the *metric* of Γ and denoted by M. For an edge e of Γ we denote by M_e the length of e in M. For any metric graph Γ, the *canonical divisor* K of Γ is the divisor on Γ with $K(x) = valency(x) - 2$ when x is a vertex of Γ and $K(x) = 0$ otherwise. When we fix the skeleton of Γ, for any metric M and divisor D on Γ we denote by $R(D)_M$ the set $R(D)$ and by $|D|_M$ the linear system $|D|$.

Remark 2. Given a metric graph $\Gamma = (V, E, M)$ and a divisor D on Γ. The divisor D is said to be *vertex-supported* if *supp*$(D) \subseteq V$. Note that *supp*(D) is always a finite set. If D is not vertex-supported, then we can refine Γ to get a new metric graph whose set of vertices is *supp*(D). Also we may assume that $|D|$ is not empty, so if D is not effective, we can consider an effective divisor $D' \in |D|$. It is obvious that $|D'| = |D|$. We shall assume from now that D is vertex-supported.

In Section 2, we present the cell complex structure of $|D|$ and introduce the *anchor cells*. We use them as landmarks to find all cells in the complex and prove a combinatorial formula (Corollary 18) for the f-vector of $|D|$. We introduce an algorithm to compute the cells of $|D|$ given Γ and D. In Section 3, we regard $R(D)$ as a tropical semi-module (convex set) and introduce an algorithm to find the extremal generators of $R(D)$, using a result in Haase et al. (2012) based on the chip-firing technique. In Section 4 we apply our algorithms to examples of trivalent graphs. Finally we raise some open problems in Section 5.

Please cite this article in press as: Lin, B. Computing linear systems on metric graphs. J. Symb. Comput. (2017), http://dx.doi.org/10.1016/j.jsc.2017.05.007