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Projective and affine symmetries and equivalences of rational curves in arbitrary dimension

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ABSTRACT

We present a new algorithm to decide whether two rational parametric curves are related by a projective transformation and detect all such projective equivalences. Given two rational curves, we derive a system of polynomial equations whose solutions define linear rational transformations of the parameter domain, such that each transformation corresponds to a projective equivalence between the two curves. The corresponding projective mapping is then found by solving a small linear system of equations. Furthermore we investigate the special cases of detecting affine equivalences and symmetries as well as polynomial input curves. The performance of the method is demonstrated by several numerical examples.

1. Introduction

It has been observed that many of the results from algebraic geometry and symbolic computation, which are available for rational curves (see the monograph by Sendra et al., 2008), are directly useful for addressing application-oriented problems in geometric modelling. In particular, several algorithms relying on symbolic computation have been designed that can solve specific problems associated with the design and the analysis of shapes.

The investigation of *singularities*, which is one of the classical topics in algebraic geometry, is clearly important for geometric design. For instance, several authors designed methods for detecting

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singularities of rational planar curves (Chen et al., 2008; Pérez-Díaz, 2007) as well as of rational space curves (Shi et al., 2013; Shi and Chen, 2010; Rubio et al., 2009) from their parametric representations, using resultants and μ-bases.

Another interesting question concerning planar rational curves is the computation and the study of their *offsets*. Some properties of offsets, such as the degree and the topological behaviour, have been investigated in Segundo and Sendra (2009) and Alcázar (2008).

Another line of research is devoted to *properties of the parameterization*. Among numerous papers on this topic, Pérez-Díaz (2006) addressed the issue of reparameterization to generate proper parameterizations, and Tabera (2011) explored the generation of optimal parameterizations in the sense of minimal coefficients.

The transformation between an *implicit and a parametric representation* of the curve is one of the most basic questions in algebraic geometry and therefore investigated in numerous publications (Rubio et al., 2006; Sederberg et al., 1997; Chen and Wang, 2002, to quote only a few).

The problem of *detecting symmetries and equivalences of curves* attracted substantial attention since it is an essential problem in Pattern Recognition, Computer Graphics and Computer Vision. For instance it is used to identify a given object with objects in a database. In Computer Graphics the knowledge about symmetries helps analyzing pictures and is applied to compression or shape completion.

Several papers on the detection of symmetries and equivalences of curves exist. In one of the earliest publications on this topic, Huang and Cohen (1996) examined affine transformations for classifying silhouettes of aircrafts, where they used B-spline moments which they approximated from a sample of points. Braß and Knauer (2004) investigated Euclidean symmetries of discrete 3-dimensional objects and proposed to apply their method to the control polygon of Bézier curves and surfaces. Lebmeir and Richter-Gebert (2008) and Lebmeir (2009) looked at Euclidean symmetries of algebraic curves given in implicit form.

More recently Alcázar (2014) and Alcázar et al. (2014a,b, 2015a,b) published a series of papers investigating the problem of symmetry and equivalence detection with respect to similarity transformations for parametric rational curves. They use the fact that the symmetry of a curve in proper parameterization is related to a rational linear transformation in the parameter domain. However, they do not consider general affine or projective equivalences. Sánchez-Reyes (2015) proposed a method for Euclidean symmetry detection of polynomial Bézier curves, based on using the Bernstein basis. For further references on symmetry detection see the introduction of Alcázar et al. (2014b).

Our paper is devoted to the detection of equivalences and symmetries of rational curves with respect to the group of *projective transformations*, of which affine ones are special cases. Similar to the works of Alcázar et al., we are looking for an exact and not approximated representation of the equivalences. We assume that exact input data is given and therefore we are able to use methods from the field of symbolic computation. More precisely, we use the monomial representations and homogeneous coordinates to derive algebraic equations that characterize the linear rational reparameterization connecting two equivalent curves.

Our method is more general than the existing ones since it handles all equivalences with respect to the full group of projective transformations and works for an arbitrary space dimension. In addition, it is easy to implement and provides good computational results for moderate degree.

The remainder of the paper is organized as follows. First we recall some basic geometric ideas and define our notation in Section 2. The main part of the paper is Section 3 where we derive a polynomial system whose solutions specify projective equivalences, characterize the size of the system and provide a comparison to a more naive approach. In Section 4 we consider some special cases, i.e., we are looking for affine equivalences and investigate the simplifications if the input curve is polynomial. Section 5 provides examples that show the simplicity of our method and give numerous further computational results. Finally in Section 6 we conclude this paper and describe some planned future work.

2. Preliminaries

We recall the different types of coordinates and recall the notions of projectively and affinely equivalent curves.

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