ARTICLE IN PRESS YJSCO:1795

Journal of Symbolic Computation ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Deciding the existence of rational general solutions for first-order algebraic ODEs

N. Thieu Vo [a,1], Georg Grasegger [b,c,1,2], Franz Winkler [b]

[a] *Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam*
[b] *Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, Austria*
[c] *Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Linz, Austria*

ARTICLE INFO

Article history:
Received 14 October 2016
Accepted 19 June 2017
Available online xxxx

MSC:
34A05
34A26
34A34
14H45
14J26
68W30

Keywords:
Ordinary differential equation
Algebraic curve
Rational parametrization
Rational general solution

ABSTRACT

In this paper, we consider the class of first-order algebraic ordinary differential equations (AODEs), and study their rational general solutions. A rational general solution contains an arbitrary constant. We give a decision algorithm for finding a rational general solution, in which the arbitrary constant appears rationally, of the whole class of first-order AODEs. As a byproduct, this leads to an algorithm for determining a rational general solution of a class of first-order AODE which covers almost all first-order AODEs from Kamke's collection. The method is based intrinsically on the consideration of the AODE from a geometric point of view. In particular, parametrizations of algebraic curves play an important role for a transformation of a parametrizable first-order AODE to a quasi-linear differential equation.

1. Introduction

A first-order algebraic differential equation (AODE) is a differential equation of the form $F(x, y, y') = 0$ for some irreducible trivariate polynomial F with coefficients in an algebraically closed

E-mail address: vongocthieu@tdt.edu.vn (N.T. Vo).

[1] The first author was a PhD student at RISC, Johannes Kepler University Linz, while working on this paper and was supported by the strategic program "Innovatives OÖ–2020" by the Upper Austrian Government.

[2] Partially supported by the Austrian Science Fund (FWF): W1214-N15, project DK11.

http://dx.doi.org/10.1016/j.jsc.2017.06.003

field $\mathbb{K}$. First-order AODEs have been studied a lot and there is a variety of solution methods for special classes. The study of such ODEs can be dated back to the work of Fuchs (1884) and Poincaré (1885). Malmquist studied the class of first-order AODEs having transcendental meromorphic solutions in Malmquist (1913), and later Eremenko (1982) revisited this problem. By using the result of Matsuda (1980) on classification of differential algebraic function fields without movable critical points, Eremenko (1998) presented a theoretical consideration on a degree bound of rational solutions.

The problem of finding closed form solutions of first-order AODEs has been considered in several papers. Kovacic (1986) solved completely the problem of computing Liouvilian solutions of a second-order linear ODEs with rational function coefficients. He also proposed an algorithm for determining all rational solutions of a Riccati equation. For the class of first-order first-degree AODEs, Carnicer (1994) studied a degree bound for algebraic solutions in the non-dicritical case. Hubert (1996) found implicit solutions by computing Gröbner bases.

We are mainly interested in rational general solutions, i.e. rational solutions which are also general solutions in the sense of Ritt (1950). Such general solutions must contain a transcendental constant. We take an algebro-geometric approach to this problem. First we neglect the differential aspect and associate to the AODE an algebraic hypersurface. In the case of an autonomous AODE of order one a rational solution of the AODE is a rational parametrization of this hypersurface. In case the hypersurface admits a rational parametrization, we have to look for a reparametrization, which would also satisfy the differential constraint; namely, that the second component of this parametrization should be the derivative of the first one. A similar reasoning is applied in the more complicated situation of non-autonomous AODEs. The algebro-geometric approach has received much attention in the last decade. Algorithms for the class of first-order autonomous AODEs have been proposed in Aroca et al. (2005); Feng and Gao (2004, 2006). The algorithm is based on the fact that if the given AODE has a rational solution, then the algebraic hypersurface obtained from the differential equation by considering the derivative as a new indeterminate is a rational curve. Applying this idea to the general class of first-order AODEs, and combining it with Fuch's theorem on first-order AODEs without movable critical points, Chen and Ma (2005) presented an algorithm for determining a special class of rational general solutions. However, their algorithm is incomplete due to two reasons: the necessary condition for the existence of the solution is not proved to be algorithmically checkable, and a good rational parametrization is required in advance. Ngô and Winkler (2010, 2011b,a) applied the algebro-geometric approach to general non-autonomous first-order AODEs. Using parametrization of algebraic surfaces, they associate to the given parametrizable AODE an associated system of algebraic equations in the parameters. This associated system is a planar rational system. In order to complete the algorithm, a degree bound for irreducible invariant algebraic curves of the planar rational system is required. The problem of finding a uniform bound for the degree of invariant algebraic curves for planar rational systems is known as the Poincaré problem. This difficult problem has been solved by Carnicer (1994), but only generically for the non-dicritical case. So the algorithm of Ngô and Winkler, although producing general rational solutions in almost all situations where such a solution exists, is still no complete decision algorithm.

So far no general algorithm for deciding the existence and, in the positive case, computing a rational general solution of first-order AODEs exists. Such a rational general solution must contain a transcendental constant. If this constant appears rationally, we speak of a strong rational solution. In this paper, we propose a full decision algorithm for taking an arbitrary first-order AODE, deciding the existence of a strong rational general solution, and in the positive case computing such a strong rational general solution. This generalizes the work of Feng and Gao (2004, 2006); Chen and Ma (2005), and Ngô and Winkler (2010, 2011b,a). More specifically, we take an algebro-geometric approach and proceed as follows: we consider the AODE $F(x, y, y') = 0$ as defining a curve over $\mathbb{K}(x)$, the field of rational functions in x over $\mathbb{K}$. I.e., we consider $y' = z$ as a new indeterminate and we view the AODE as an algebraic equation $F(x, y, z) = 0$ defining an algebraic curve in the affine plane $\mathbb{A}^2(\overline{\mathbb{K}(x)})$. We prove that in order for the AODE to have a strong rational general solution, in which the transcendental constant appears rationally, the algebraic curve must be of genus 0. We also prove that over $\mathbb{K}(x)$ every rational curve has an optimal parametrization with coefficients in $\mathbb{K}(x)$, i.e., without algebraically extending the coefficient field. Such an optimal parametrization allows us to transform the given AODE to an associated AODE which is easier to solve. Consequently, we derive a full decision