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Effective identifiability criteria for tensors and polynomials

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ABSTRACT

A tensor T, in a given tensor space, is said to be h-identifiable if it admits a unique decomposition as a sum of h rank one tensors. A criterion for h-identifiability is called effective if it is satisfied in a dense, open subset of the set of rank h tensors. In this paper we give effective h-identifiability criteria for a large class of tensors. We then improve these criteria for some symmetric tensors. For instance, this allows us to give a complete set of effective identifiability criteria for ternary quintic polynomials. Finally, we implement our identifiability algorithms in Macaulay2.

1. Introduction

A tensor rank decomposition of a tensor T, lying in a given tensor space over a field k, is an expression of the type

$$T = \lambda_1 U_1 + \ldots + \lambda_h U_h \tag{1.1}$$

where the U_i's are linearly independent rank one tensors, $\lambda_i \in k^*$, and k is either the real or complex field. The *rank* of T, denoted by $\mathrm{rank}(T)$, is the minimal positive integer h such that T admits a decomposition as in (1.1).

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Tensor decomposition problems and techniques are of relevance in both pure and applied mathematics. For instance, tensor decomposition algorithms have applications in psycho-metrics, chemometrics, signal processing, numerical linear algebra, computer vision, numerical analysis, neuroscience and graph analysis (Kolda and Bader, 2009; Comon and Mourrain, 1996; Comon et al., 2008; Landsberg and Ottaviani, 2015; Massarenti and Raviolo, 2013, 2014). In pure mathematics tensor decomposition issues naturally arise in constructing and studying moduli spaces of all possible additive decompositions of a general tensor into a given number of rank one tensors (Dolgachev, 2004; Dolgachev and Kanev, 1993; Massarenti and Mella, 2013; Massarenti, 2016; Ranestad and Schreyer, 2000; Takagi and Zucconi, 2011).

We say that a tensor rank one decomposition has the *generic identifiability property* if the expression (1.1) is unique, up to permutations and scaling of the factors, on a dense open subset of the set of tensors admitting an expression as in (1.1). This uniqueness property is useful in several application, we refer to Chiantini et al. (2017a) and Hauenstein et al. (2016) for an account.

We would like to mention that in Hauenstein et al. (2016), using new numerical methods and higher order flattenings the authors discovered several new cases of identifiability, and furthermore they proposed a conjecture on generic identifiability.

Given a tensor rank one decomposition of length h as in (1.1) the problem of *specific identifiability* consists in proving that such a decomposition is unique. Following Chiantini et al. (2017a) we call an algorithm for specific identifiability *effective* if it is sufficient to prove identifiability on a dense open subset of the set of tensors admitting a decomposition as in (1.1). Therefore, an algorithm is effective if its constraints are satisfied generically, in other words if the same algorithm proves generic identifiability as well.

In this paper we consider symmetric tensors, mixed skew-symmetric tensors, and mixed symmetric tensors. The corresponding rank one tensors are parametrized respectively by Veronese varieties, Segre–Grassmann varieties, and Segre–Veronese varieties. We provide h-identifiability effective criteria for these spaces, under suitable numerical assumptions on h. Our algorithms are based on the existence of suitable flattenings of a given tensor admitting a decomposition as in (1.1). We would like to stress that we do not need to know an explicit decomposition but just the fact that such a decomposition exists.

Recall that the border rank $\underline{\operatorname{rank}}(T)$ of a tensor T is the smallest integer $r > 0$ such that T is in the Zariski closure, in the tensor space where T belongs, of the set of tensors of rank r. In particular $\underline{\operatorname{rank}}(T) \leq \operatorname{rank}(T)$. Roughly speaking, our methods require that suitable linear spaces, defined in terms of flattenings, intersect the relevant varieties parametrizing rank one tensors in a zero-dimension scheme of a given length. Such a zero dimensional scheme is not required to be reduced and then our criteria can be applied also in border rank identifiability problems, see Remark 3.7.

Symmetric tensors can also be interpreted as homogeneous polynomials. By rephrasing (1.1) in the symmetric case we say that a polynomial rank one decomposition of a homogeneous degree d polynomial $F \in k[x_0, \dots, x_n]_d$ is an expression of the type

$$F = \lambda_1 L_1^d + \dots + \lambda_h L_h^d \tag{1.2}$$

where L_i are linearly independent degree 1 polynomials, $\lambda_i \in k^*$, and k is either the real or complex field. Let $h(n,d)$ be the minimum integer such that a general $F \in k[x_0, \dots, x_n]_d$ admits a decomposition as in (1.2). The number $h(n,d)$ has been determined in Alexander and Hirschowitz (1995) and $h(n,d)$-identifiability very seldom holds (Mella, 2006, 2009; Galuppi and Mella, 2017). Indeed, by Galuppi and Mella (2017, Theorem 1) a general polynomial $F \in k[x_0, \dots, x_n]_d$ is $h(n,d)$-identifiable only in the following cases:

- $n = 1$, $d = 2m+1$, $h(n,d) = m$ (Sylvester, 1904),
- $n = d = 3$, $h(3,3) = 5$ (Sylvester, 1904),
- $n = 2$, $d = 5$, $h(2,5) = 7$ (Hilbert, 1888).

In Theorem 3.8 we provide effective h-identifiability criteria for these polynomials and combined with the previous results this furnishes a complete set of identifiability criteria for these, and few more,

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