Journal of Symbolic Computation ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Symbolic Computation

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A near-optimal subdivision algorithm for complex root isolation based on the Pellet test and Newton iteration

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ARTICLE INFO

Article history:
Received 14 January 2016
Accepted 17 January 2017
Available online xxxx

Keywords:
Root finding
Root isolation
Approximate arithmetic
Certified computation
Complexity analysis
Complex roots
Subdivision methods

ABSTRACT

We describe a subdivision algorithm for isolating the complex roots of a polynomial $F \in \mathbb{C}[x]$. Given an oracle that provides approximations of each of the coefficients of F to any absolute error bound and given an arbitrary square $\mathcal{B}$ in the complex plane containing only simple roots of F, our algorithm returns disjoint isolating disks for the roots of F in $\mathcal{B}$.

Our complexity analysis bounds the absolute error to which the coefficients of F have to be provided, the total number of iterations, and the overall bit complexity. It further shows that the complexity of our algorithm is controlled by the geometry of the roots in a near neighborhood of the input square $\mathcal{B}$, namely, the number of roots, their absolute values and pairwise distances. The number of subdivision steps is near-optimal. For the *benchmark problem*, namely, to isolate all the roots of a polynomial of degree n with integer coefficients of bit size less than τ, our algorithm needs $\tilde{O}(n^3 + n^2\tau)$ bit operations, which is comparable to the record bound of Pan (2002). It is the first time that such a bound has been achieved using subdivision methods, and independent of

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[2] Supported by IMPECS (Indo-German Max Planck Center for Computer Science), "Exact Geometric Computation in Nonlinear Problems".

[3] Supported by ANR/DFG project STU 483/2-1 SMArT.

[4] Supported by National Science Foundation Grant #CCF-1423228.

http://dx.doi.org/10.1016/j.jsc.2017.03.009

Please cite this article in press as: Becker, R., et al. A near-optimal subdivision algorithm for complex root isolation based on the Pellet test and Newton iteration. J. Symb. Comput. (2017), http://dx.doi.org/10.1016/j.jsc.2017.03.009

divide-and-conquer techniques such as Schönhage's splitting circle technique.

Our algorithm uses the quadtree construction of Weyl (1924) with two key ingredients: using Pellet's Theorem (1881) combined with Graeffe iteration, we derive a "soft-test" to count the number of roots in a disk. Using Schröder's modified Newton operator combined with bisection, in a form inspired by the quadratic interval method from Abbot (2006), we achieve quadratic convergence towards root clusters. Relative to the divide-conquer algorithms, our algorithm is quite simple with the potential of being practical. This paper is self-contained: we provide pseudo-code for all subroutines used by our algorithm.

1. Introduction

The computation of the roots of a univariate polynomial is one of the best studied problems in the areas of computer algebra and numerical analysis, nevertheless there are still a number of novel algorithms presented each year; see McNamee and Pan (2012; 2013), McNamee (2002; 2007), Pan (1997) for an extensive overview. One reason for this development is undoubtedly the great importance of the problem, which results from the fact that solutions for many problems from mathematics, engineering, computer science, or the natural sciences make critical use of univariate root solving. Another reason for the steady research is that, despite the huge existing literature, there is still a large discrepancy between methods that are considered to be efficient in practice and those that achieve good theoretical bounds. For instance, for computing all complex roots of a polynomial, practitioners typically use Aberth's, Weierstrass–Durand–Kerner's and QR algorithms. These iterative methods are relatively simple as, in each step, we only need to evaluate the given polynomial (and its derivative) at certain points. They have been integrated in popular packages such as MPSolve (Bini and Fiorentino, 2000; Bini and Robol, 2014) or `eigensolve` (Fortune, 2002), regardless of the fact that their excellent empirical behavior has not been entirely verified in theory. In contrast, there exist algorithms (Emiris et al., 2014; Mehlhorn et al., 2015; Pan, 2002) that achieve near-optimal bounds with respect to asymptotic complexity; however, implementations of these methods do not exist. The main reason for this situation is that these algorithms are quite involved and that they use a series of asymptotically fast subroutines (see Pan, 2002, p. 702). In most cases, this rules out a self-contained presentation, which makes it difficult to access such methods, not only for practitioners but also for researchers working in the same area. In addition, for an efficient implementation, it would be necessary to incorporate a sophisticated precision management and many implementation tricks. Even then, there might still be a considerable overhead due to the extensive use of asymptotically fast subroutines, which does not show up in the asymptotic complexity bounds but is critical for input sizes that can be handled on modern computers.

In this paper, we aim to resolve the above described discrepancy by presenting a subdivision algorithm for complex root isolation, which we denote by ℂIsolate. For our method, we mainly combine simple and well-known techniques such as the classical quad-tree construction by Weyl (1924), Pellet's Theorem (Rahman and Schmeisser, 2002), Graeffe iteration[5] (Best, 1949; Householder, 1959), and Schröder's modified Newton operator (Schröder, 1870). In addition, we derive bounds on its theoretical worst-case complexity matching the best bounds currently known for this problem; see Section 1.1 for more details. Hence, we hope that our contribution will finally bring

[5] Following Householder (1959), the method should not be attributed to Gräffe only. In fact, it was developed independently by Dandelin in 1826 and Gräffe in 1837. Also, Lobachevsky's contribution from 1834 is significant as it contains the key idea of the approach. For simplicity, we stick to the commonly used designation "Graeffe iteration" even though it seems to be more correct to call the method "Dandelin–Lobachevsky–Graeffe method".

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