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Small partial Latin squares that embed in an infinite group but not into any finite group ☆

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ABSTRACT

Suppose that Y_1, Y_2, Y_3 are finite sets and $P \subseteq Y_1 \times Y_2 \times Y_3$. We say that P embeds in a group G if there exist injective maps $\phi_i : Y_i \to G$ for $i = 1, 2, 3$ such that $\phi_1(y_1)\phi_2(y_2) = \phi_3(y_3)$ for each $(y_1, y_2, y_3) \in P$. Hirsch and Jackson asked for the cardinality of the smallest P that embeds in some infinite group but not into any finite group. We prove that the answer to their question is 12. Moreover, we show that there are 50 examples of cardinality 12, up to equivalence, and each of them embeds in the (infinite) Baumslag group $G = \langle a, b \mid b = [b, b^a] \rangle$. Our proof uses computations to answer questions about finitely presented groups which are known to be algorithmically undecidable in general.

1. Introduction and results

We define a ***partial Latin square*** (PLS) P to be a matrix in which some cells may be empty and in which each filled cell contains one symbol from an underlying alphabet $\Lambda(P)$, such that no symbol occurs more than once within any row or column. The ***size*** of P is the number of filled cells in P, and the ***shape*** $\mathcal{S}(P)$ of P is the set of filled cells in P. We avoid degeneracies by insisting that each row and column contains at least one filled cell, and that each element of $\Lambda(P)$ appears at least once in P. A side-effect is that our PLS need not be square matrices. In some references, PLS are defined to be square matrices and to have at least as many rows as there are symbols. To achieve these properties

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it is always possible to add empty rows and/or empty columns to our PLS. Allowing any finite number of empty rows and columns would create some technical nuisances but would not materially affect any of the questions we are interested in solving.

Let $P = [P_{i,j}]$ be an $m \times n$ PLS. An ***embedding*** $\phi\colon P \hookrightarrow G$ of P into a group G is a triple (ϕ_1, ϕ_2, ϕ_3) of injective maps

$$\phi_1\colon \{1, 2, \ldots, m\} \to G, \quad \phi_2\colon \{1, 2, \ldots, n\} \to G, \quad \phi_3\colon \Lambda(P) \to G,$$

such that $\phi_1(i)\phi_2(j) = \phi_3(P_{i,j})$ for all $(i, j) \in \mathcal{S}(P)$. We refer to $r_i = \phi_1(i)$ for $i = 1, \ldots, m$ as the ***row labels***, and $c_j = \phi_2(j)$ for $j = 1, \ldots, n$ as the ***column labels***. Intuitively, P embeds in G if a copy of P can be found within the Cayley table of G (in the subtable which has row and column labels $r_1, \ldots, r_m$ and $c_1, \ldots, c_n$, respectively). We refer to Cavenagh and Wanless (2009) for a discussion of applications of embeddings of PLS in groups and connections with linear algebra and topological graph theory. Note that embedding in groups is a special type of completion problem for PLS. See Keedwell and Dénes (2015, Chap. 3) for an introduction to the rich literature of such problems.

A useful notion of equivalence for PLS is obtained by converting each PLS P into a set of triples $\mathcal{T}(P) = \{(r, c, P_{r,c}) : (r, c) \in \mathcal{S}(P)\}$. We say that two PLS are from the same ***species*** if they produce the same set of triples, modulo uniform permutation of the 3 coordinates in the triples, and relabelling within each coordinate. The property of having an embedding in a given group is a species invariant Cavenagh and Wanless (2009, Lem. 1), so in this paper it will suffice to consider one representative from each species of PLS.

Hirsch and Jackson (2012, Ex. 3.7) gave an example of a PLS of size 29 that can be embedded in an infinite group, but not in any finite group. They noted that smaller examples exist, and posed the question of what the smallest possible size of such a PLS is. The purpose of this paper is to answer this question by proving:

Theorem 1. *There are* 50 *species of PLS of size* 12 *that can be embedded in an infinite group, but not in any finite group. No PLS of smaller size has the same property.*

One of the 50 PLS of size 12 is detailed in Lemma 7. The remainder of the paper discusses our computational approach which proves Theorem 1. These computations involve manipulating finite presentations for groups to test, for example, whether a group is finite or whether two group elements are equal. These questions are equivalent to solving the "*word problem*" and hence, in full generality, are well known to be algorithmically undecidable, cf. Holt et al. (2005, §5) or Robinson (1982, p. 54). So we were somewhat fortunate to find methods which solved the instances of these problems that we needed to solve.

For recent work related to this paper we refer to Wanless and Webb (2017), which identifies all smallest PLS that (a) do not embed into any group, (b) embed into a group but do not embed into any abelian group, or (c) embed into an abelian group but do not embed into any cyclic group. Each of those classes contains small PLS that can be found with simpler methods than we employ in the present paper.

2. The presentation defined by a PLS

The next lemma (cf. Cavenagh and Wanless 2009, Lem. 2) is an easy observation and shows that we lose no generality by assuming that the associated row and column labels of an embedding satisfy $r_1 = c_1 = 1$, the identity of the group.

Lemma 2. *If a PLS P can be embedded into a group G, then it can be embedded with row labels $r_1, \ldots, r_m \in G$ and column labels $c_1, \ldots, c_n \in G$ that satisfy $r_1 = c_1 = 1$, the identity in G.*

Proof. If $(\phi_1, \phi_2, \phi_3)\colon P \hookrightarrow G$ is an embedding with associated row labels $r_1, \ldots, r_m$ and column labels $c_1, \ldots, c_n$, then $(\phi'_1, \phi'_2, \phi'_3)\colon P \hookrightarrow G$ defined by $\phi'_1(r) = r_1^{-1}\phi_1(r)$, $\phi'_2(c) = \phi_2(c)c_1^{-1}$, and $\phi'_3(e) = r_1^{-1}\phi_3(e)c_1^{-1}$ is an embedding with the required property. $\square$