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# A smoothness test for higher codimensions ☆

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ABSTRACT

Based on an idea in Hironaka's proof of resolution of singularities, we present an algorithm for determining smoothness of algebraic varieties. The algorithm is inherently parallel and does not involve the calculation of codimension-sized minors of the Jacobian matrix of the variety. We also describe a hybrid method which combines the new method with the Jacobian criterion, thus making use of the strengths of both approaches. We have implemented all algorithms in the computer algebra system SINGULAR. We compare the different approaches with respect to timings and memory usage. The test examples originate from questions in algebraic geometry, where the use of the Jacobian criterion is impractical due to the number and size of the minors involved.


## 1. Introduction

In classical algebraic geometry, explicit constructions of algebraic varieties with prescribed properties play an important role, for example, for existence and unirationality results in moduli problems, see e.g. Neves and Papadakis (2009), Schreyer (2013). In many situations, the aim is to construct a *smooth* variety satisfying certain properties. For example, when considering a family of algebraic curves with a given Hilbert polynomial, the arithmetic genus of the curve is determined. The geometric genus, however, differs from the arithmetic genus by the delta invariant, which measures the

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singularities of the curve. Hence the presence of singularities affects the geometric genus, leading to a different topological type of the curve. Passing to the next dimension, recall that algebraic surfaces have been classified in the Enriques–Kodaira classification, see e.g. Barth et al. (2004). Especially for surfaces of general type, a full understanding of the moduli spaces and explicit constructions of canonical rings are still lacking. Although there are various techniques to construct surfaces with prescribed invariants, see e.g. Bauer et al. (2006), Mendes Lopes and Pardini (2008), Neves and Papadakis (2009), the constraint of smoothness often requires testing and, in practice, this turns out to be a fundamental obstacle. It is thus of equal importance to have a test which is both fast in determining smoothness and non-smoothness.

The standard method for testing smoothness is the Jacobian criterion, (cf. any textbook on computational algebraic geometry, e.g. Greuel and Pfister, 2008, Cor. 5.6.14). Given an equidimensional affine algebraic variety $X = V(I) \subseteq \mathbb{A}^n$ of codimension $c$ defined by an ideal $I = \langle f_1, \ldots, f_s \rangle \subseteq k[x_1, \ldots, x_n]$ over an algebraically closed field $k$, we compute the dimension of the vanishing locus of the Jacobian ideal $J$, which is generated by $c \times c$-minors of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ on $X$. This can be done by computing a Gröbner basis of the ideal $I + J$. However, the number $\binom{n}{c} \cdot \binom{s}{c}$ of minors can be very large, and the Gröbner basis determines the complete scheme structure of the singular locus of $X$, which is not required to check smoothness. As a result, this approach will be rather inefficient and often even impractical.

In this paper, we describe an algorithm for determining smoothness, which is based on an idea from Hironaka's famous proof of resolution of singularities. The (implicitly stated) termination criterion provides a smoothness criterion, which does not require computation of the $c \times c$-minors of the Jacobian matrix. The key idea behind this smoothness test is the fact that each non-singular variety is locally a complete intersection. That is, it can be covered by suitable open subsets, on each of which we truly see a complete intersection. Such a covering can be computed without too much effort and, in relevant examples, our approach significantly extends the limits of practicability of the Jacobian criterion. In addition, the construction is inherently parallel with only minimal communication overhead. In fact, application of the criterion can in some cases be faster than computing a single minor of the Jacobian matrix. Our algorithm is implemented in the library `smoothtst.lib` (Böhm and Frühbis-Krüger, 2016) for the computer algebra system SINGULAR (Decker et al., 2015).

This paper is organized as follows: In Section 2, we extract the relevant part of Hironaka's smoothness criterion from the constructive, simplified desingularization approach of Bravo et al. (2005). In Section 3, we turn this criterion into an algorithm and then, in Section 4, refine it into a hybrid approach, which combines it with the use of the Jacobian criterion in smaller codimension. Section 5 recalls some settings and constructions from algebraic geometry whose practical use often requires an efficient smoothness test, and presents explicit examples of such constructions. These are then used in Section 6 to compare our new approaches with the standard technique based on the Jacobian criterion.

## 2. Hironaka's smoothness criterion

In 1964, Hironaka proved existence of resolutions of singularities in characteristic zero (Hironaka, 1964). He introduced standard bases to achieve this goal. Though his proof is non-constructive in certain parts, standard bases are by no means the only algorithmic considerations introduced there. The termination criterion, which he uses, provides a smoothness criterion that does not involve the computation of the ideal of codimension-sized minors of the Jacobian matrix.

For this article, we assume $k$ to be an algebraically closed field of characteristic zero unless explicitly stated otherwise. The general line of arguments is still valid, if we drop the condition on the field to be algebraically closed, but everything needs to be stated with significantly more care.

**Definition 2.1** *(Hironaka (1964))*. Let $(X, 0) \subset (\mathbb{A}^n_k, 0)$ be a germ with defining ideal $I_{X,0} \subset k\{\underline{x}\} := k\{x_1, \ldots, x_n\}$ generated by $f_1, \ldots, f_s$ in the ring of convergent power series, and assume that the