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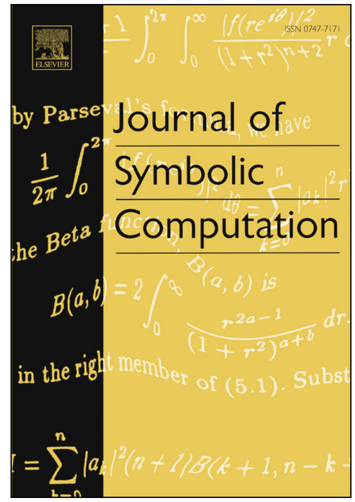

Embedding algorithms and applications to differential equations

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EMBEDDING ALGORITHMS AND APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Abstract. Algorithms for embedding certain types of nilpotent subalgebras in maximal subalgebras of the same type are developed, using methods of real algebraic groups. These algorithms are applied to determine non-conjugate subalgebras of the symmetry algebra of the wave equation, which in turn are used to determine a large class of invariant solutions of the wave equation. The algorithms are also illustrated for the symmetry algebra of a classical system of differential equations considered by Cartan in the context of contact geometry.

1. Introduction

One of the main applications of Lie algebras is to find solutions of differential equations, by reduction of order, or by using conjugacy classes of its subalgebras to find invariant solutions. The method of invariant solutions goes back to [Lie2, Ch. X]. This method is also explained in detail in the books of Ibragimov [Ib1, Ch 9], Ibragimov [Ib2], Bluman [Bl, B2] and Olver [Ol, Ch 3].

The Lie theoretic input in this method is a list of conjugacy classes of subalgebras of dimension depending on the order of the equation. A detailed structure of the symmetry algebra is also useful in finding linearizing coordinates for linearizable equations.

It is our experience, based on [ADGM], that if the algebras are not chosen appropriately, they are practically useless, because a preliminary step is to find their invariants and there is no algorithmic procedure to do that. However, if the subalgebras are constructed from the geometry of the space on which one is studying a given equation, for example by embedding translations or scalings in maximal subalgebras, the characteristics of the subalgebras obtained are manageable.

The principal aim of this note is to give algorithms for embedding given abelian and solvable algebras of certain types in maximal subalgebras of the same type, using standard commands of Maple.

The precise types of the subalgebras are given in the algorithms constructed below.

Maple is able to find the Cartan decomposition as well as root space decompositions for semisimple algebras of fairly high dimensions. The algorithms it uses are based on the fundamental papers of Rand, Winternitz and Zassenhaus [RWZ], of de Graff [dG], and of Dietrich, Faccin and de Graaf [DFG] and Ian Anderson [An]. The recent book of Šnobl-Winternitz [SW] gives a detailed account of some of these algorithms.

Derksen, Jeandel and Koiran [DJK] have also developed algorithms for computing the Zariski closures of linear solvable Lie groups and the algorithms in this paper reduce the computation of Zariski closures of linear groups to those of abelian subgroups. The algorithms given in this paper are based on results of Mostow [Mo] on real algebraic groups; see a recent account of the

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