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Singularities of plane rational curves via projections

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A B S T R A C T

We consider the parameterization $\mathbf{f} = (f_0 : f_1 : f_2)$ of a plane rational curve C of degree n, and we study the singularities of C via such parameterization. We use the projection from the rational normal curve $C_n \subset \mathbb{P}^n$ to C and its interplay with the secant varieties to C_n. In particular, we define via $\mathbf{f}$ certain 0-dimensional schemes $X_k \subset \mathbb{P}^k$, $2 \le k \le (n-1)$, which encode all information on the singularities of multiplicity $\ge k$ of C (e.g. using X_2 we can give a criterion to determine whether C is a cuspidal curve or has only ordinary singularities). We give a series of algorithms which allow one to obtain information about the singularities from such schemes.

1. Introduction

The study of plane rational curves is quite classical in algebraic geometry, and it is also an interesting subject for applications, for example it is very relevant in Computer Aided Design (CAD). Since rational curves are the ones that can be parameterized, it is quite of interest to get information on the curve from its parameterization (implicit equation, structure of singularities, e.g. see Song et al. (2007), Sendra et al. (2008), Cox et al. (1998), Cox et al. (2013)). In this paper we tackle the problem

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of determining the singularities of a plane rational curve from its parameterization. This is a problem which has been much treated in the literature: see the beautiful work Cox et al. (2013), where the syzygies of the ideal generated by the polynomials giving the parameterization are used in order to determine the singularities of the curve and their structure (multiplicity, branches, infinitely near other singularities).

This idea has been developed also in Song et al. (2007) and Chen et al. (2008), where "μ-bases" are exploited for the parameterized curve. We used this approach in a previous paper, Bernardi et al. (2016), in order to find how a plane curve could be viewed as a projection of a rational curve contained in a rational normal scroll.

In the present paper we describe the structure of singular points by using the parameterization, but from a different point of view with respect to the one of Bernardi et al. (2016). In order to study the singularities of a plane rational curve C of degree n, we use the fact that the parameterization of C defines a projection $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, which is generically one-to-one from the rational normal curve $C_n \subset \mathbb{P}^n$ onto its image, and $\pi(C_n) = C \subset \mathbb{P}^2$. If P is a singular point of multiplicity m of $C \subset \mathbb{P}^2$, then there is an $(m-1)$-dimensional m-secant space H to C_n in $\mathbb{P}^n$ such that $\pi(H) = P$. The center of projection of π is a $(n-3)$-linear space Π, and $\Pi \cap H$ has to be $(m-2)$-dimensional, in order to have that $\pi(H)$ is a point. We have that $H \cap C_n$ (and $H \cap \Pi$) contains all the information about the singularity P of C (multiplicity, branches, infinitely near points, e.g. see Moe (2008)); the problem is how to extract this information from these data.

Our strategy here is to consider, for $k = 2, \dots, n-1$, the spaces $\mathbb{P}^k \cong \mathbb{P}(K[s,t]_k)$ that parameterize $\sigma_k(C_n)$, the k-secant variety of $(k-1)$-dimensional k-secant spaces to C_n and their intersection with the center of projection Π, which is determined by the parameterization of C. Such study yields to considering certain 0-dimensional schemes, $X_k \subset \mathbb{P}^k$, which parameterize the k-secant $(k-1)$-spaces that get contracted to a point by the projection π, so that they encode all the information on the singularities of C. For example, we can use the scheme X_2 to give simple necessary and sufficient conditions for the curve C to be a cuspidal one or to have only ordinary singularities (see Proposition 3.1).

This approach stems out from a study of the so-called Poncelet varieties associated to rational curves (see Ilardi et al. (2009)); in a previous paper (see Bernardi et al. (2015b)) we considered the singularities of Poncelet surfaces in order to determine the existence of triple points on C. Here that approach has been generalized and potentially covers all kind of singular points on C. The interplay between secant varieties $\sigma_k(C_n)$ of rational normal curves and 0-dimensional subschemes of the space $\mathbb{P}^k$ parameterizing the $\mathbb{P}^{k-1}$ k-secant spaces of C_n has also been studied by the authors in Bernardi et al. (2011).

Our choice has been to give our main results in the form of algorithms that can be used to study a given rational curve C, for example with the help of programs as CoCoA (Capani et al., 1995), Macaulay2 (Grayson and Stillman, 2009) or Bertini (Bates et al., 2013). Our Algorithm 1 allows one to compute the number N of singularities of C and also the number N_k of singularities of multiplicity k, for $k = 2, \dots, n-1$. We also give a variation of Algorithm 1 (cf. Algorithm 1.1) that allows one to compute, for each multiplicity, the number of singular points with given number of branches and multiplicity of each branch. Our Algorithm 2 computes the numbers $N, N_2, \dots, N_{n-1}$ too, but it also gives the ideal of each subset $\mathrm{Sing}_k(C) \subset \mathrm{Sing}(C)$ given by the points with multiplicity k. Eventually, Algorithm 3 gives the (maybe approximated) coordinates of the points in $\mathbb{P}^1$ which, via the parameterization $\mathbf{f}$ of C, are the preimages of the singular points of C. This allows one to compute the coordinates of the singular points of C by applying $\mathbf{f}$ to such points.

Although algorithms determining the structure of plane curves singularities do exist (see Chen et al. (2008), Pèrez-Dìaz (2007), Cox et al. (2013)), we think that our algorithms can be a useful tool, also used together with the existing ones, since their approach to the problem is different, and their behavior on specific curves can be of different effectiveness.

The paper is organized as follows: in the next section we give all the preliminary notions and define the schemes X_k which will be our crucial tool to study $\mathrm{Sing}(C)$. In Section 3 we give the algorithms mentioned above. Section 4 is dedicated to the study of curves with only double points. In this case we give a criterion (Theorem 4.3) to describe, via the projection $\pi|_{C_n} : C_n \to C$, which kind of singularity a double point can be. This result is interesting in itself, since usually this description

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