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Cylindrical algebraic decomposition using local projections

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ABSTRACT

We present an algorithm which computes a cylindrical algebraic decomposition of a semialgebraic set using projection sets computed for each cell separately. Such local projection sets can be significantly smaller than the global projection set used by the Cylindrical Algebraic Decomposition (CAD) algorithm. This leads to reduction in the number of cells the algorithm needs to construct. A restricted version of the algorithm was introduced in Strzeboński (2014). The full version presented here can be applied to quantified formulas and makes use of equational constraints. We give an empirical comparison of our algorithm and the classical CAD algorithm.

1. Introduction

A semialgebraic set is a subset of $\mathbb{R}^n$ which is a solution set of a system of polynomial equations and inequalities. Computation with semialgebraic sets is one of the core subjects in computer algebra and real algebraic geometry. A variety of algorithms have been developed for real system solving, satisfiability checking, quantifier elimination, optimization and other basic problems concerning semialgebraic sets (Collins, 1975; Basu et al., 2006; Caviness and Johnson, 1998; Chen et al., 2009; Dolzmann et al., 1998; Grigoriev and Vorobjov, 1988; Hong and Din, 2012; Loos and Weispfenning, 1993; Renegar, 1992; Tarski, 1951; Weispfenning, 1993). Every semialgebraic set can be represented as a finite union of disjoint cells bounded by graphs of algebraic functions. The Cylindrical Algebraic Decomposition (CAD) algorithm (Collins, 1975; Caviness and Johnson, 1998;

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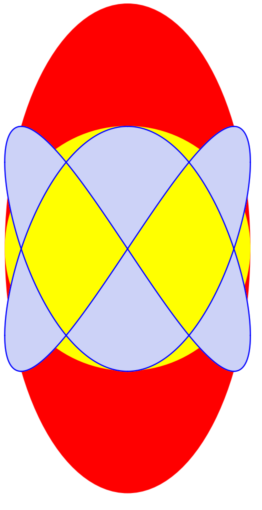
Strzeboński, 2006) can be used to compute a cell decomposition of any semialgebraic set presented by a quantified system of polynomial equations and inequalities. An alternative method of computing cell decompositions is given in Chen et al. (2009). Cell decompositions computed by the CAD algorithm can be represented directly (Strzeboński, 2006, 2010; Brown, 2003) as cylindrical algebraic formulas (CAF; see the next section for a precise definition). CAF representation of a semialgebraic set A can be used to decide whether A is nonempty, to find the minimal and maximal values of the first coordinate of elements of A, to generate an arbitrary element of A, to find a graphical representation of A, to compute the volume of A, to compute multidimensional integrals over A (Strzeboński, 2000), or to solve polynomial equations and inequalities over A (Strzeboński, 2012). Boolean combinations of semialgebraic sets represented by CAFs can be computed using the algorithm given in Strzeboński (2010).

The CAD algorithm takes a, possibly quantified, system of polynomial equations and inequalities and constructs a cell decomposition of its solution set. The algorithm consists of two phases. The projection phase finds a set of polynomials whose roots are sufficient to describe the cell boundaries. The lifting phase constructs a cell decomposition, one dimension at a time, subdividing cells at all roots of the projection polynomials. However, some of these subdivisions may be unnecessary, either because of the geometry of the roots or because of the Boolean structure of the input system. In this paper we propose an algorithm which combines the two phases. It starts with a sample point and constructs a cell containing the point on which the input system has a constant truth value. Projection polynomials used to construct the cell are selected based on the structure of the system at the sample point. Such a local projection set can often be much smaller than the global projection set used by the CAD algorithm. The idea to use such locally valid projections was first introduced in Jovanovic and de Moura (2012), in an algorithm to decide the satisfiability of systems of real polynomial equations and inequalities. It was also used in Brown (2013), in an algorithm to construct a single open cell from a cylindrical algebraic decomposition. An earlier version of this paper appeared in Strzeboński (2014). Here we extend the algorithm to handle equational constraints and quantified systems and include a wider range of examples in the empirical comparison.

Example 1. Find a cylindrical algebraic decomposition of the solution set of $S = f_1 < 0 \vee (f_2 \le 0 \wedge f_3 \le 0)$, where $f_1 = 4x^2 + y^2 - 4$, $f_2 = x^2 + y^2 - 1$, and $f_3 = 16x^6 - 24x^4 + 9x^2 + 4y^4 - 4y^2$.



The solution set of S is equal to the union of the open ellipse $f_1 < 0$ and the intersection of the closed disk $f_2 \le 0$ and the set $f_3 \le 0$ bounded by a Lissajous curve. As can be seen in the picture, the set is equal to the open ellipse $f_1 < 0$. The CAD algorithm uses a projection set consisting of the discriminants and the pairwise resultants of f_1, f_2, and f_3. It computes a cell decomposition of the

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