Journal of Symbolic Computation 76 (2016) 158–199

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Associated primes of spline complexes

Michael DiPasquale [1]

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma, United States

ARTICLE INFO

Article history:
Received 25 October 2014
Accepted 23 December 2015
Available online 19 January 2016

MSC:
primary 13C05
secondary 13P25, 13C10

Keywords:
Polyhedral spline
Polytopal complex
Localization
Hilbert function

ABSTRACT

The spline complex whose top homology is the algebra of mixed splines over a fan was introduced by Schenck–Stillman as a variant of a complex of Billera. In this paper we analyze the associated primes of homology modules of this complex. In particular, we show that all such primes are linear. We give two applications to computations of dimensions. The first is a computation of the third coefficient of the Hilbert polynomial of the algebra of mixed splines over a fan, including cases where vanishing is imposed along arbitrary codimension one faces of the boundary of the fan, extending computations by Geramita–Schenck in the simplicial case and McDonald–Schenck in the polytopal case. The second is a description of the fourth coefficient of the Hilbert polynomial of the algebra of mixed splines over simplicial fans. We use this to re-derive a result of Alfeld–Schumaker–Whiteley on the generic dimension of C^1 tetrahedral splines in large degree and indicate via an example how this description may be used to give the fourth coefficient in particular non-generic configurations.

1. Introduction

Let $\mathcal{P}$ be a subdivision of a region in $\mathbb{R}^n$ by convex polytopes. $C^r(\mathcal{P})$ denotes the set of piecewise polynomial functions (splines) on $\mathcal{P}$ that are continuously differentiable of order r. Study of the spaces $C^r(\mathcal{P})$ is a fundamental topic in approximation theory and numerical analysis (see de Boor, 2001) while within the past decade geometric connections have been made between $C^0(\mathcal{P})$ and equivariant

E-mail address: mdipasq@okstate.edu.

[1] Author supported by the National Science Foundation grant DMS 0838434 “EMSW21MCTP: Research Experience for Graduate Students.”

http://dx.doi.org/10.1016/j.jsc.2016.01.004
0747-7171/

cohomology rings of toric varieties (Payne, 2006). Splines are currently used in a wide variety of other applications such as computer aided geometric design (CAGD) (Farin, 1997) and isogeometric analysis (Cottrell et al., 2009).

A central problem in spline theory is to determine the dimension of (and a basis for) the vector space $C^r_d(\mathcal{P})$ of splines whose restriction to each facet of $\mathcal{P}$ has degree at most d. The spaces $C^r_d(\Delta)$ for simplicial complexes Δ in $\mathbb{R}^2$ and $\mathbb{R}^3$ have been well-studied using Bernstein–Bézier methods by Alfeld and co-authors (Alfeld and Schumaker, 1987, 1990, 2008; Alfeld et al., 1993). A signature result appears in Alfeld and Schumaker (1990), which gives a dimension formula for $C^r_d(\mathcal{P})$ when $d \geq 3r+1$ and $\mathcal{P}$ is a generic simplicial complex.

An algebraic approach to the dimension question was pioneered by Billera (1988) using homological and commutative algebra. He introduces a chain complex $\mathcal{R}/\mathcal{I}$, whose top homology is the spline algebra. Using a computation due to Whiteley (1991b), he deduces the dimension of C^1 splines over generic triangulations $\Delta \subset \mathbb{R}^2$, solving a conjecture of Strang (1973). Schenck and Stillman (1997) use a similar chain complex $\mathcal{R}/\mathcal{J}$ to compute the dimension of $C^r_d(\Delta)$, $\Delta \subset \mathbb{R}^2$, for $d \gg 0$. McDonald and Schenck (2009), building on work of Rose (1995, 2004) on dual graphs, extend this method to give the dimension $C^r_d(\mathcal{P})$ of splines over a polytopal subdivision $\mathcal{P} \subset \mathbb{R}^2$ for $d \gg 0$.

The results of this paper are as follows. Working in the context of fans $\Sigma \subset \mathbb{R}^{n+1}$, we introduce the notation $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma']$ for the spline complex, where $\Sigma' \subset \Sigma$ is a subfan. This is well-suited to describing the spline complexes that arise from imposing vanishing along codimension one faces of the boundary, in such a way that topological contributions are made explicit. Using the notion of a lattice fan, first introduced by DiPasquale (2014a), we describe localizations of the spline complex $\mathcal{R}/\mathcal{J}[\Sigma, \Sigma']$. We then prove Theorem 5.5, which identifies the associated primes of the homology modules of the spline complex as linear primes arising from the hyperplane arrangement of affine spans of codimension one faces, and Theorem 5.7, which identifies more precisely the associated primes of minimal possible codimension (this is a slight extension of Schenck, 2012, Theorem 2.6).

We give two applications of these theorems to computations of dimension of the space $\dim C^\alpha(\Sigma)$. In Section 8, we derive the third coefficient of the Hilbert polynomial of the graded algebra $C^\alpha(\Sigma)$ of mixed splines on the polyhedral fan $\Sigma \subset \mathbb{R}^{n+1}$, where vanishing may be imposed along arbitrary codimension one faces of the boundary of Σ (Corollary 8.3). This result draws on two papers of Schenck, together with Geramita and McDonald, where the third coefficient is computed in the simplicial mixed smoothness case and the polytopal uniform smoothness case, respectively (Geramita and Schenck, 1998; McDonald and Schenck, 2009); however no boundary conditions are imposed in either of these papers. The computation in Section 8 also clarifies certain topological contributions to the third coefficient.

In Section 9, we describe the fourth coefficient of the Hilbert polynomial of the graded algebra $C^\alpha(\widehat{\Delta})$, where $\Delta \subset \mathbb{R}^3$ is a simplicial complex (Proposition 9.1). For the case of uniform smoothness $r=1$ we give an explicit form for the Hilbert polynomial of $C^1(\widehat{\Delta})$ in Corollary 9.6. We conclude by using Corollary 9.6 together with earlier work of Whiteley to recover the computation (for $d \gg 0$) of Alfeld–Schumaker–Whiteley on the dimension of $C^1_d(\widehat{\Delta})$ for generic $\Delta \subset \mathbb{R}^3$ (Alfeld et al., 1993).

2. Polytopal complexes and fans

In this section we introduce polytopal complexes and polyhedral fans, which are the underlying objects over which we define splines. References for the material in this section are Ziegler (1995) for polytopal complexes and Cox et al. (2011) for fans. We will also need to use on occasion the notion of a *cell complex*; the basics of cell complexes may be found in Chapter 0 of Hatcher (2002).

Definition 2.1. Fix a vector space $\mathbb{R}^n$ of dimension n. A *convex polytope* is the convex hull of a finite set V of points, namely

$$\sigma = \operatorname{conv}(V) = \{\sum_{v \in V} \lambda_v v \mid \lambda_v \geq 0 \in \mathbb{R} \text{ and } \sum_{v \in V} \lambda_v = 1\}.$$