Journal of Symbolic Computation 72 (2016) 21–54

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Continued fraction real root isolation using the Hong root bound

George E. Collins

ARTICLE INFO

Article history:
Received 13 April 2013
Accepted 15 November 2014
Available online 19 December 2014

Keywords:
Polynomial roots
Real roots
Root isolation
Continued fractions
Maximum computing time
Root bounds
Dominance
Algorithm analysis

ABSTRACT

An investigation of the codominance maximum computing time of the continued fractions method (CF) for isolation of the real roots of a squarefree integral polynomial when applied to the two-parameter family of polynomials $A_{a,n}(x) = x^n - 2(ax^2 - (a+2)x + 1)^2$, with $n \geq 5$ and $a \geq 1$. These polynomials have two roots, r_1 and r_2, in the interval $(0, 1)$, with $|r_1 - r_2| < a^{-n}$. It is proved that for these polynomials the maximum time required by CF to isolate those two close roots would be codominant with $n^5(\ln a)^2$ even if an "ideal" root bound were available and either the Horner method or the Budan method is used for translations. It is proved that if a power-of-two Hong root bound is used by CF to determine translation amounts then the time required to isolate the two close roots is dominated by $n^6(\ln a)$ if a multiplication-free Budan translation method is used. Computations reveal that the Hong root bound is surprisingly effective when applied to the transformed polynomials that arise, engendering a minimum efficiency conjecture. It is proved that if the conjecture is true then the time to isolate the two close roots is dominated by $n^5(\ln a)^2$. There is also evidence for a maximum efficiency conjecture. The two conjectures together, if true, make it likely that this time is codominant with $n^5(\ln a)^2$.

1. Introduction

This paper is a sequel to Collins and Krandick (2012), where it was proved that the computing time of the continued fractions positive root isolation method (CF) for the polynomials $x^n - 2(x^2 - 3x + 1)^2$, $n \geq 5$, dominates n^5. Here we generalize that result, by considering the two-parameter family of

E-mail address: gcollins8@charter.net.

http://dx.doi.org/10.1016/j.jsc.2014.11.001
0747-7171/

polynomials $A_{a,n}(x) = x^n - 2(ax^2 - (a+2)x + 1)$, $a \geq 1$ and $n \geq 5$. We adopt the notation of Collins and Krandick (2012) for polynomial transformations. Specifically, $\mathrm{T}_h(A(x)) = A(x+h)$, $\mathrm{T} = \mathrm{T}_1$ and $\mathrm{R}(A(x)) = x^n A(1/x)$ where $n = \deg(A)$, and $\mathrm{H}_d(A(x)) = A(dx)$. Also $\mathrm{r}(x) = x^{-1}$, $\mathrm{t}_h(x) = x+h$ and $\mathrm{t} = \mathrm{t}_1$. We will use the symbols $\preceq$ and $\succeq$ for dominance, $\sim$ for codominance, as in Collins (1974). Throughout the paper $a \geq 1$ and $n \geq 5$ are implicit hypotheses.

In Section 2 we prove that $A_{a,n}$ is irreducible, that it has two roots, r_1 and r_2, in $(0,1)$, and that all other roots are outside of the circle with center at the origin and radius 1. We also prove that r_1 and r_2 are on opposite sides of r, the root of $ax^2 + (a+2)x - 1$ in $(0,1)$, and differ from r by at most $r^{n/2+1} < a^{-n/2-1}$. Finally we prove that the continued fraction of r is $[0, a+1, a, a, \ldots]$.

In Section 3 we define an infinite sequence of polynomials C_i with $C_0 = A_{a,n}$, $\tilde{C}_i = \mathrm{R}(C_i)$, $C_1 = \mathrm{T}_{a+1}(\tilde{C}_0)$ and $C_{i+1} = \mathrm{T}_a(\tilde{C}_i)$ for $i \geq 1$, only a finite number of which, N, are computed. We show that $N \geq \lfloor \frac{n}{4} \rfloor$.

In Section 4 we define two integer sequences, d_i and e_i, and derive expressions for the coefficients of C_i in terms of them. For $i \leq \lfloor \frac{n}{4} \rfloor$ CF also computes $\mathrm{T}(C_i)$, and we show, using these expressions, that the time CF requires just to perform these translations dominates either $n^5(\ln a)$ or $n^5(\ln a)^2$, depending on which of three classical translation methods is used.

In Section 5 we use the d_i and e_i sequences to obtain an upper bound on the coefficients of the C_i and also to obtain a linear upper bound on N, thereby proving that N is codominant with n.

In Section 6 we express the bounds on the C_i coefficients as functions of a and i and then apply the results of Sections 4 and 5 to prove that the time for all $\mathrm{T}(C_i)$ translations is codominant with $n^5 \ln a$.

In Section 7 we analyze the time that would be required to compute each C_{i+1} from $\tilde{C}_i$ if one had a fictional “ideal” root bound method that, without cost, delivers the floor function of the least positive root of any polynomial. Applied to each $\tilde{C}_i$ the result would be a. We prove that the time to compute $C_{i+1} = \mathrm{T}_a(\tilde{C}_i)$ for $0 \leq i \leq N$ is dominated by $n^5(\ln a)^2$ and is codominant with $n^5(\ln a)^2$ if the translations are performed by either Horner’s method or Budan’s method.

In Section 8 we consider CFHLB, the CF method equipped with a subalgorithm that outputs an integer lower bound for the positive roots of any transformed polynomial. Specifically CFHLB utilizes the Hong root bound (Hong, 1998), in a power-of-two version. We prove that the computing time of CFHLB to isolate the two close roots of the polynomials $A_{a,n}$ in $(0,1)$ is dominated by $n^6(\ln a)$ provided that multiplication-free Budan translations by powers of two are used and by $n^6(\ln a)^2$ otherwise.

We exhibit in Section 9 the surprisingly effective performance of the Hong root bound on the transformed polynomials that arise and we base on this evidence a minimum efficiency conjecture that implies a computing time that is dominated by $n^5(\ln a)^2$ for CFHLB to isolate the two close roots of $A_{a,n}$ in $(0,1)$ provided that the multiplication-free Budan translation method is used. We also find evidence for a maximum efficiency conjecture. The two conjectures together imply that the number of translations needed by CFHLB to compute C_{i+1} from $\tilde{C}_i$ is codominant with $\ln a$.

In Section 10 we discuss several problems left unsolved by this paper.

2. The two close roots

The continued fractions method separately isolates the roots in $(0,1)$ and the roots in $(1,\infty)$. In this section we prove that $A_{a,n}$ is irreducible, has exactly two roots in $(0,1)$, and that all its other roots are outside of the circle of radius 1 centered at the origin of the complex plane. We prove that the two roots in $(0,1)$, r_1 and r_2, satisfy $r - h < r_1 < r < r_2 < r + h$, where $h = r^{n/2+1}$ and r is the root of $B_a(x) = ax^2 - (a+2)x + 1$ in $(0,1)$. We also prove that the continued fraction of r is $[0, a+1, a, a, a, \ldots]$.

Theorem 1. *Let $B(x) = ax^2 - (a+2)x + 1$. $B_a(x)$ has two positive roots, one in $(0,1)$ and one in $(1,\infty)$.*

Proof. $B_a(0) = 1$, $B_a(1) = -1$ and $B_a(\infty) = \infty$. □