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A difference ring theory for symbolic summation ☆

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ABSTRACT

A summation framework is developed that enhances Karr's difference field approach. It covers not only indefinite nested sums and products in terms of transcendental extensions, but it can treat, e.g., nested products defined over roots of unity. The theory of the so-called $R\Pi\Sigma^*$-extensions is supplemented by algorithms that support the construction of such difference rings automatically and that assist in the task to tackle symbolic summation problems. Algorithms are presented that solve parameterized telescoping equations, and more generally parameterized first-order difference equations, in the given difference ring. As a consequence, one obtains algorithms for the summation paradigms of telescoping and Zeilberger's creative telescoping. With this difference ring theory one gets a rigorous summation machinery that has been applied to numerous challenging problems coming, e.g., from combinatorics and particle physics.

1. Introduction

In his pioneering work M. Karr (1981, 1985) introduced a very general class of difference fields, the so-called $\Pi\Sigma$-fields, in which expressions in terms of indefinite nested sums and products can be represented. In particular, he developed an algorithm that decides constructively if for a given

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expression $f(k)$ represented in a $\Pi\Sigma$-field $\mathbb{F}$ there is an expression $g(k)$ represented in the field $\mathbb{F}$ such that the telescoping equation (anti-difference)

$$f(k) = g(k+1) - g(k) \tag{1}$$

holds. If such a solution exists, one obtains for an appropriately chosen $a \in \mathbb{N}$ the identity

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a). \tag{2}$$

His algorithms can be viewed as the discrete version of Risch's integration algorithm; see Risch (1969), Bronstein (1997). In the last years the $\Pi\Sigma$-field theory has been pushed forward. It is now possible to obtain sum representations, i.e., right hand sides in (2) with certain optimality criteria such as minimal nesting depth (Schneider, 2008, 2010c), minimal number of generators in the summands (Schneider, 2004c) or minimal degrees in the denominators (Schneider, 2007b). For the simplification of products see Schneider (2005c), Abramov and Petkovšek (2010). We emphasize that exactly such refined representations give rise to more efficient telescoping algorithms worked out in Schneider (2010b, 2015).

A striking application is that Karr's algorithm and all the enhanced versions can be used to solve the parameterized telescoping problem (Schneider, 2000, 2010a): for given indefinite nested product-sum expressions $f_1(k), \dots, f_n(k)$ represented in $\mathbb{F}$, find constants $c_1, \dots, c_n$, free of k and not all zero, and find $g(k)$ represented in $\mathbb{F}$ such that

$$g(k+1) - g(k) = c_1\, f_1(k) + \cdots + c_n\, f_n(k) \tag{3}$$

holds. In particular, this problem covers Zeilberger's creative telescoping paradigm (Zeilberger, 1991) for a bivariate function $F(m,k)$ by setting $f_i(k) = F(m+i-1,k)$ with $i \in \{1,\dots,n\}$ and representing these $f_i(k)$ in $\mathbb{F}$. Namely, if one finds such a solution, one ends up at the recurrence

$$g(m,b+1) - g(m,a) = c_1 \sum_{k=a}^{b} f(m,k) + \cdots + c_n \sum_{k=a}^{b} f(m+n-1,k).$$

In a nutshell, one cannot only treat indefinite summation but also definite summation problems. In this regard, also recurrence solvers have been developed where the coefficients of the recurrence and the inhomogeneous part can be elements from a $\Pi\Sigma$-field (Bronstein, 2000; Schneider, 2005d; Abramov et al., in preparation). All these algorithms generalize and enhance substantially the (q-)hypergeometric and holonomic toolbox (Abramov, 1971; Gosper, 1978; Zeilberger, 1990, 1991; Petkovšek, 1992; Paule, 1995; Petkovšek et al., 1996; Paule and Riese, 1997; Bauer and Petkovšek, 1999; Chyzak, 2000; Kauers and Paule, 2011; Koutschan, 2013) in order to rewrite definite sums to indefinite nested sums. For details on these aspects we refer to Schneider (2014).

Besides all these sophisticated developments, e.g., within the summation package `Sigma` (Schneider, 2007c), there is one critical gap which concerns all the developed tools in the setting of difference fields: Algebraic products, like

$$(-1)^k = \prod_{i=1}^{k}(-1), \quad (-1)^{\binom{k+1}{2}} = \prod_{i=1}^{k}\prod_{j=1}^{i}(-1), \quad (-1)^{\binom{k+2}{3}} = \prod_{i=1}^{k}\prod_{j=1}^{i}\prod_{k=1}^{j}(-1), \dots \tag{4}$$

cannot be expressed in $\Pi\Sigma$-fields, which are built by a tower of transcendental field extensions. Even worse, the objects given in (4) introduce zero-divisors, like

$$(1-(-1)^k)(1+(-1)^k) = 0 \tag{5}$$

which cannot be treated in a field or in an integral domain. In applications these objects occur rather frequently as standalone objects or in nested sums (Ablinger et al., 2011, 2013). It is thus a fundamental challenge to include such objects in an enhanced summation theory.