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Formalizing polygonal knot origami ☆

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We present computer-assisted construction of regular polygonal knots by origami. The construction is completed with an automated proof based on algebraic methods. Given a rectangular origami or a finite tape, of an adequate length, we can construct the simplest knot by three folds. The shape of the knot is made to be a regular pentagon if we fasten the knot tightly without distorting the tape. We perform the analysis of the knot fold further formally towards the automated construction and verification. In particular, we show the construction and proof of regular pentagonal and heptagonal knots. We employ a software tool called Eos (e-origami system), which incorporates the extension of Huzita's basic fold operations for construction, and Gröbner basis computation for proving. Our study yields more mathematical rigor and in-depth results about the polygonal knots.

1. Introduction

Given a rectangular origami, i.e. sheet of folding paper of adequate length, we can construct the simplest knot by performing three times of folds. The shape of the knot is made to be a regular pentagon if we fasten the knot tightly without distorting the origami as shown in Fig. 1. The method is so simple that it should be known since the age of early human civilization. However, only from

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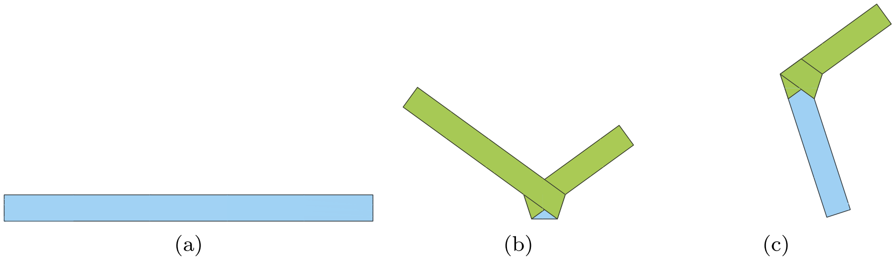

Fig. 1. A regular pentagon obtained by the simplest knot of three folds.

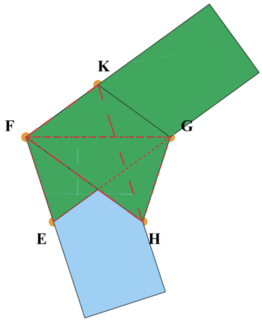

Fig. 2. Regular pentagonal knot and overlapping isosceles triangles.

the middle of the 20th century mathematical investigations of the knot fold construction started to appear (e.g. see, Brunton, 1961; Sakaguchi, 1982; Wells, 1991).

In this paper we conduct further a formal analysis of the knot fold for the realization of the computer-assisted construction and verification. The knot fold is decomposed into a sequence of more basic folds, each creating an isosceles triangle as shown in Fig. 2. Three overlapping and congruent isosceles triangles make a pentagonal knot. By superposing the isosceles triangles carefully, we can construct a regular $2n(n \geq 2) + 1$-gonal knot in general. If we relax the rigidity of the constructed shape, but yet without breaking the origami, we can make more kinds of regular n-gons, some with a hole in the center. We show those properties, focusing on the constructions of regular pentagonal and heptagonal knots.

Knot fold is interesting because of its familiarity to everyone, of its common use in everyday life and of its simple principle. In everyday life we make knots by various materials such as strings, ropes, cloths and papers, and we can imagine various forms of knots depending on them. To avoid possible misunderstandings arising from the variety of the used substances and to focus on essential properties of knots, mathematical aspects of knots have been studied deeply since the middle of the 18th century. In this paper we are interested in the concrete shapes created by knotting the paper tape. Hence, we are less concerned with the topological and combinatorial aspects of knots, but are more with the methods of the construction. Furthermore, since a paper tape of a rectangular shape can be constructed from a square sheet of paper (i.e. origami) by repeated folds, we will consider a rectangular origami from the outset. Subsequently, we simply call it a *tape* or an *origami*.

Knot fold is based on physical constraints, i.e. the rigidity and foldability of the paper, unlike Huzita's basic folds that are based on the observable incidence relations over inductively defined points and lines, and on their superpositions (Huzita, 1989). From the origami point of view, the knot fold can be seen as a variant of multi-fold, where we perform folds along multiple fold lines simultaneously (Alperin and Lang, 2009). A general multi-fold is difficult to perform precisely by hand, whereas we will see the knot fold by hand retains a certain degree of precision as far as the constructed shape is concerned.