Journal of Symbolic Computation 69 (2015) 129–158

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Beyond polynomials and Peano arithmetic—automation of elementary and ordinal interpretations

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ARTICLE INFO

Article history:
Received 31 January 2014
Accepted 25 June 2014
Available online 29 September 2014

Keywords:
Term rewriting
Termination
Automation
Ordinals

ABSTRACT

Kirby and Paris (1982) proved in a celebrated paper that a theorem of Goodstein (1944) cannot be established in Peano arithmetic. We present an encoding of Goodstein's theorem as a termination problem of a finite rewrite system. Using a novel implementation of algebras based on ordinal interpretations, we are able to automatically prove termination of this system, resulting in the first automatic termination proof for a system whose derivational complexity is not multiple recursive. Our method can also cope with the encoding by Touzet (1998) of the battle of Hercules and Hydra as well as a (corrected) encoding by Beklemishev (2006) of the Worm battle, two further systems which have been out of reach for automatic tools, until now. Based on our ideas of implementing ordinal algebras we also present a new approach for the automation of elementary interpretations for termination analysis.

1. Introduction

Since the beginning of the millennium there has been much progress regarding automated termination tools for rewrite systems.[1] Despite the many different techniques that have been developed, it seems that (terminating) TRSs which admit very long derivations are out of reach even for the most

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[1] http://www.termination-portal.org/.

http://dx.doi.org/10.1016/j.jsc.2014.09.033
0747-7171/

powerful tools. This is not surprising since many base methods induce rather small upper bounds on the derivational complexity, which is a function that bounds the length of the longest possible derivation (rewrite sequence) by the size of its starting term. Hofbauer and Lautemann (1989) have shown that polynomial interpretations are limited to double exponential derivational complexity. They further showed that the derivational complexity of a rewrite system compatible with the Knuth–Bendix order (KBO) cannot be bounded by a primitive recursive function. Later, Lepper (2001) established the Ackermann function as an upper bound for KBO, whereas Weiermann (1995) proved a multiple recursive upper bound for the lexicographic path order (LPO). More recently, Moser and Schnabl (2011) and Schnabl (2012) have studied upper bounds on the complexity when using these base methods in the dependency pair framework. Although dependency pairs significantly increase termination proving power, from the viewpoint of derivational complexity the limit is still multiple recursive. This has led to the conjecture (Schnabl, 2012, Conjecture 6.99) that for any system whose termination can be proved automatically by modern tools the length of its derivations can be bounded by a multiple recursive function (in the size of the starting terms).

Ordinals have been used in termination arguments for many decades (e.g., Turing, 1949; Gentzen, 1936). In fact ordinals are essential to prove termination of the battle of Hercules and Hydra (also due to Kirby and Paris, 1982), or the sequences associated with Goodstein's theorem since these derivations cannot be bounded by a multiple recursive function (Cichon, 1983). Although TRS encodings of the Hydra battle are known for many years (e.g., by Touzet, 1998), they could so far not be handled by automatic termination tools, witnessing Schnabl's conjecture. Indeed a successful implementation of ordinals for automatic termination proofs is still lacking. Very recently, Urban and Miné (2014) presented an approach to conclude termination of imperative programs by inferring ordinal-valued ranking functions. Here ordinals are essential to handle nondeterminism, though only ordinals below $\omega^{\omega^{\omega}}$ are involved and hence the ranking functions are still multiple recursive. The theorem prover Vampire uses ordinal numbers (see Kovács et al., 2011, Section 7) in its implementation of KBO but only for weights of predicate symbols. Since these symbols occur only at the root of atomic expressions no ordinal arithmetic is needed but only comparison of ordinals.

In this article we first encode the computation of Goodstein sequences (see Theorem 9) as a rewrite system $\mathcal{G}$ such that termination of $\mathcal{G}$ implies Goodstein's theorem. Since these sequences cannot be bounded by a multiple recursive function, this also holds for the derivational complexity of $\mathcal{G}$. After presenting this motivating example, we discuss automation of a termination criterion based on ordinal interpretations which is capable of proving $\mathcal{G}$ terminating, thereby overcoming the limitations alleged by the above conjecture. Our implementation can also cope with Touzet's encoding (Touzet, 1998) of the battle of Hercules and Hydra, as well as a (corrected) encoding of the Worm battle (Beklemishev, 2006).

Automation of ordinal interpretations is challenging since ordinal arithmetic does, e.g., not satisfy commutativity. Hence in contrast to polynomial interpretations terms do not evaluate to expressions of a canonical shape. We tackle this deficiency by introducing approximations which yield expressions of a special shape. Approximations (albeit less involved) have already been used for polynomial interpretations with negative (Hirokawa and Middeldorp, 2004; Fuhs et al., 2007) or irrational (Zankl and Middeldorp, 2010) coefficients. In preliminary work Zankl et al. (2012) and Winkler et al. (2012) already used ordinal domains to increase automatic termination proving power. However, in Zankl et al. (2012) the focus is on string rewriting and the interpretation functions have a very limited shape to avoid ordinal arithmetic. As a consequence the method is limited to systems with at most multiple *exponential* derivational complexity. Similarly, Winkler et al. (2012) use ordinal domains for generalized KBO, again for string rewriting only. In the respective implementation, function symbol weights are moreover below ω^{ω}. We anticipate that our treatment of arithmetic for ordinals up to ϵ_0 could improve some of the results from Kovács et al. (2011), Winkler et al. (2012), and Urban and Miné (2014).

Lescanne (1995) proposed elementary functions for proving (AC-)termination but his implementation is limited to checking the orientation of rules for *given* interpretations. Lucas (2009) considers so-called linear elementary interpretations (LEIs) of the shape $A(\bar{x}) + B(\bar{x})^{C(\bar{x})}$ where $A(\bar{x})$, $B(\bar{x})$, and $C(\bar{x})$ are linear polynomials. Furthermore, he proposes an approach based on rewriting, constraint logic programming (CLP), and constraint satisfaction problems (CSPs) to also *find* suitable interpreta-