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On decomposable semigroups and applications

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ABSTRACT

In this work we develop a framework to decrease the time complexity of well-known algorithms to compute the generator sets of a semigroup ideal by using the Hermite normal form. We introduce idea of decomposable semigroups, which fulfills that the computation of its ideal can be achieved by separately calculating over smaller semigroups, products of the decomposition. Our approach does not only decrease the time complexity of the problem, but also allows using parallel computational techniques. A combinatorial characterization of these semigroups is obtained and the concept of decomposable variety is introduced. Finally, some applications and practical results are provided.

0. Introduction

Let $\Bbbk$ be a field, and let S be a finitely generated subsemigroup of an Abelian group with $A = \{a_1, \ldots, a_n\}$ a fixed system of generators of S. The semigroup ideal of S is the binomial ideal (see Herzog, 1970)

$$I_S = \left\langle X^\alpha - X^\beta \;\middle|\; \sum_{i=1}^n \alpha_i a_i = \sum_{i=1}^n \beta_i a_i \right\rangle \subset \Bbbk[X_1, \ldots, X_n]$$

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with $\Bbbk[X_1,\ldots,X_n]$ the S-graduated polynomial ring where the S-degree a_i is assigned to the indeterminate X_i. Thus, the S-degree of $X^\alpha = X_1^{\alpha_1}\cdots X_n^{\alpha_n}$ is $\sum_{i=1}^n \alpha_i a_i \in S$ (see Miller and Sturmfels, 2005 for further details).

The study of semigroup ideals began in the last third of the 20th century (see Herzog, 1970 and the references therein) and has become an important research area due to its connections with other scientific fields such as Algebraic Statistic, Coding Theory, Combinatorics, Integer Programming, Toric Geometry, etc. One of the most prominent topic in this research area is the improvement of computational aspects for checking ideal properties and for computing systems of generators (see Briales et al., 1998; Hemmecke and Malkin, 2009; Sturmfels, 1996; Vigneron-Tenorio, 1999 and the references therein).

The main purpose of this work is to present a preprocessing method of polynomial time complexity that improves the computation of the ideal for a decomposable semigroup. A semigroup $S=\langle A\rangle$ is *decomposable* if it is the direct sum of some proper subsemigroups $S_i=\langle A_i\rangle$ and then

$$I_S = I_{S_1} + \cdots + I_{S_t}, \tag{1}$$

with $A=\coprod_{i=1}^t A_i$, otherwise it is irreducible. Note that each ideal I_{S_i} is included in a polynomial ring with as many variables as elements belong to A_i (strictly less than n). Theorem 9 states that a semigroup is decomposable if and only if the Hermite normal form of the matrix associated to its system of generator has a special form. Therefore it can be detected whether a semigroup is decomposable and obtain its decomposition by using an algorithm of polynomial time complexity (see Micciancio and Warinschi, 2001). Since the time complexity for computing a semigroup ideal is simply exponential in the number of variables (see Sturmfels, 1993), for decomposable semigroups the exponent of the time complexity can be reduced to the maximum of the number of variables of I_{S_i} with $i=1,\ldots,t$. In summary, the time complexity is reduced by using a preprocessing algorithm of polynomial time complexity and once we know the decomposition of the semigroup, the computation of its generating set can be done in parallel to further reduce its computation time.

Decomposition (1) allows us to take advantages of computation of some generating sets of I_S. In this way, each type of generating set of I_S (Gröbner bases, Graver bases, universal Gröbner bases, Markov bases and universal Markov bases) can be obtained directly from the corresponding generating set of the ideals I_{S_i} in (1). In addition, some properties of decomposable semigroups and their ideals can be studied from the semigroups and ideals that appear in their decompositions. We prove that I_S is a complete intersection if and only if I_{S_i} is a complete intersection for every i, and that a decomposable semigroup is a gluing if and only if at least one of the subsemigroups of its decomposition is a gluing.

The above purely algebraic decomposition has a geometric interpretation. Since every ideal defines an algebraic variety, decomposition (1) can be used to obtain a decomposition of the variety and that allows us to introduce the concept of *decomposable variety*. These varieties admit parametrizations with simple formulations.

If S satisfies that $S\cap(-S)=\{0\}$, Theorem 18 gives a combinatorial characterization of decomposable semigroups by using the simplicial complex introduced in Eliahou (1983) and the simplicial complex used in Briales et al. (1998) (see Ojeda and Vigneron-Tenorio, 2010a for further details).

The contents of this paper are organized as follows. In Section 1 some definitions, notation and some known results are introduced. In Section 2 it is showed how the Hermite normal form can be used to obtain a diagonalization of a matrix which is called HNF-diagonal matrix. Algorithm 7 is the key that allows us to easily compute the decomposition of a semigroup. In Section 3 decomposable semigroups are characterized by using the HNF-diagonalization of a matrix associated to S. In Section 4 decomposable varieties are introduced and a method to obtain simple parametrizations is presented. In Section 5 some relationships between decomposable semigroups (and their ideals) and the semigroups (and their ideals) appearing in the decomposition are showed. The goal of Section 6 is to obtain a combinatorial characterization of decomposable semigroups. In Section 7, our results are illustrated with an example from Algebraic Statistics. Finally in Section 8, HNF-decomposition is used to get bases of the ideals of some statistical models and random matrices. The time of computation is used to make comparisons and to draw conclusions on the improvement achieved by the HNF-decomposition.