Knowledge-Based Systems 94 (2016) 114–123

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

A new distance-based total uncertainty measure in the theory of belief functions

Yi Yang [a], Deqiang Han [b,*]

[a] SKLSVMS, School of Aerospace, Xi'an Jiaotong University, Xi'an 710049, China
[b] Institute of Integrated Automation, Xi'an Jiaotong University, Xi'an 710049, China

ARTICLE INFO

Article history:
Received 27 July 2015
Revised 22 October 2015
Accepted 20 November 2015
Available online 27 November 2015

Keywords:
Belief functions
Evidence theory
Uncertainty measure
Belief interval
Distance of interval

ABSTRACT

The theory of belief functions is a very important and effective tool for uncertainty modeling and reasoning, where measures of uncertainty are very crucial for evaluating the degree of uncertainty in a body of evidence. Several uncertainty measures in the theory of belief functions have been proposed. However, existing measures are generalizations of measures in the probabilistic framework. The inconsistency between different frameworks causes limitations to existing measures. To avoid these limitations, in this paper, a new total uncertainty measure is proposed directly in the framework of belief functions theory without changing the theoretical frameworks. The average distance between the belief interval of each singleton and the most uncertain case is used to represent the total uncertainty degree of the given body of evidence. Numerical examples, simulations, applications and related analyses are provided to verify the rationality of our new measure.

1. Introduction

Dempster–Shafer evidence theory (DST) [1], also called the theory of belief functions, has been widely used in many applications related to uncertainty modeling and reasoning, e.g., information fusion [2], pattern classification [3,4], clustering analysis [5], fault diagnosis [6], and multiple attribute decision-making (MADM) [7,8].

In the theory of belief functions, there are two types of uncertainty including the discord (or conflict or randomness) [9] and the non-specificity [10], hence the ambiguity [11]. Various kinds of measures for these two types of uncertainty and the total uncertainty including both two types were proposed. The measures of discord include the discord measure [9], the strife [9], the confusion [12], etc; the measures of non-specificity include Dubois and Prade's definition [10] generalized from the Hartley entropy [13] in the classical set theory, Yager's definition [14], and Korner's definition [15], etc. The most representative total uncertainty measures are the aggregated measure (*AU*) [16] and the ambiguity measure (*AM*) [11].

In essential, no matter *AU* or *AM*, they are the generalization of Shannon entropy [17] in probability theory, but not the direct definition in the framework of the theory of belief functions. That is, they pick up a probability according to some criteria or constraints established based on the given body of evidence (BOE), and then calculate the corresponding Shannon entropy of the probability to indirectly depict the degree of uncertainty for the given body of evidence. As mentioned in the related references [11,18], *AU* and *AM* have their own limitations. For example, they are insensitive to the change of BBA. These limitations to some extent are related to the inconsistency [19,20] between the framework of the theory of belief functions and that of the probability theory. Therefore, to avoid the limitations of the traditional definitions for the uncertainty measure, in our work, a new total uncertainty measure is proposed directly in the framework of the theory of belief functions without the switching between different frameworks. We analyze the belief interval and conclude that belief intervals carry both the randomness part and the imprecision part (non-specificity) in the uncertainty incorporated in a BOE. Thus, it is feasible to define a total uncertainty measure for a BOE. Given a BOE, the distance between the belief interval of each singleton and the most uncertain interval [0, 1] is used for constructing the degree of total uncertainty. The larger the average distance, the smaller the degree of uncertainty. Since there is no switch of theoretical frameworks, our new definition has desired properties including the ideal value range and the monotonicity. Furthermore, the uncertainty measure can provide more rational results when compared with the traditional ones, which can be supported by experimental results and related analyses.

The rest of this paper is organized as follows. Section 2 provides the brief introduction of the theory of belief functions. Commonly used uncertainty measures in the theory of belief functions are

* Corresponding author. Tel.: +86 13119115975.
E-mail addresses: jiafeiyy@mail.xjtu.edu.cn (Y. Yang), deqhan@gmail.com, deqhan@mail.xjtu.edu.cn (D. Han).

http://dx.doi.org/10.1016/j.knosys.2015.11.014
0950-7051/

introduced in Section 3. Some drawbacks of the available total uncertainty measures including *AM* and *AU* are also pointed out in Section 3. In Section 4, a new total uncertainty measure is proposed. Some desired properties and related analyses on the new proposed definition are also provided. Experiments and simulations are provided in Section 5 to show the rationality of our proposed total uncertainty measure. An application of the new total uncertainty measure on feature evaluation is provided in Section 6. Section 7 concludes this work.

2. Basics of the theory of belief functions

The basic conception in the theory of belief functions [1] is the frame of discernment (FOD), whose elements are mutually exclusive and exhaustive, representing what we concern. $m: 2^{\Theta} \rightarrow [0, 1]$ is called a basic belief assignment (BBA) defined over an FOD Θ if it satisfies

$$\sum\nolimits_{A \subseteq \Theta} m(A) = 1,\ m(\emptyset) = 0 \tag{1}$$

When $m(A) > 0$, A is called a focal element. A BBA is also called a mass function. The set of all the focal elements denoted by $\mathfrak{F}$ and their corresponding mass assignments constitute a body of evidence (BOE) $(\mathfrak{F}, m)$.

The belief function (*Bel*) and plausibility function ((*Pl*)) are defined as

$$Bel(A) = \sum\nolimits_{B \subseteq A} m(B) \tag{2}$$

$$Pl(A) = \sum\nolimits_{A \cap B \neq \emptyset} m(B) \tag{3}$$

The belief function $Bel(A)$ represents the justified specific support for the focal element (or proposition) A, while the plausibility function $Pl(A)$ represents the potential specific support for A. The length of the belief interval $[Bel(A), Pl(A)]$ is used to represent the degree of imprecision for A.

Two independent BBAs $m_1(\cdot)$ and $m_2(\cdot)$ can be combined using Dempster's rule of combination as [1]

$$m(A) = \begin{cases} 0, \pounds\ A = \emptyset \\ \dfrac{\sum\limits_{A_i \cap B_j = A} m_1(A_i) m_2(B_j)}{1 - \sum\limits_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j)}, \pounds A \neq \emptyset \end{cases} \tag{4}$$

There are still some other alternative combination rules. See [21] for details. The theory of belief functions has been criticized for its validity [19,20,22–25]. It is not a successful generalization of the probability theory, i.e., there exists inconsistency [20] between the framework of the theory of belief functions and that of the probability theory.

3. Uncertainty measures in the theory of belief functions

In the theory of belief functions, there are two types of uncertainty including the discord (or the conflict or the randomness) and the non-specificity, hence ambiguity [11].

3.1. Measure of discord in belief function

Measures of discord are for describing the randomness (or discord or conflict) in a BOE [11]. Available definitions are listed below. Although with various names, they are all for the discord part of the uncertainty in a BOE.

(1) Confusion measure [12]

$$Conf(m) = -\sum_{A \subseteq \Theta} m(A) \log_2 (Bel(A)) \tag{5}$$

(2) Dissonance measure [14]

$$Diss(m) = -\sum_{A \subseteq \Theta} m(A) \log_2 (Pl(A)) \tag{6}$$

(3) Discord measure [26]

$$Disc(m) = -\sum_{A \subseteq \Theta} m(A) \log_2 \left[1 - \sum_{B \subseteq \Theta} m(B) \frac{|B - A|}{|B|} \right] \tag{7}$$

(4) Strife measure [9]

$$Stri(m) = -\sum_{A \subseteq \Theta} m(A) \log_2 \left[1 - \sum_{B \subseteq \Theta} m(B) \frac{|A - B|}{|A|} \right] \tag{8}$$

Note that all these definitions can be considered as Shannon entropy-alike measures. More detailed information on these measures can be found in [9].

3.2. Measures for non-specificity in belief function

Non-specificity [15,27] means two or more alternatives are left unspecified and represents an imprecision degree. It only focuses on those focal elements with cardinality larger than one. Non-specificity is a special uncertainty type in the framework of belief functions theory when compared with the probabilistic framework. Some non-specificity measures [10,14,15] were proposed. The most commonly used non-specificity definition is [10]

$$NS(m) = \sum_{A \subseteq \Theta} m(A) \log_2 |A| \tag{9}$$

It can be regarded as a generalized Hartley measure [13] from the classical set theory. When the BBA $m(\cdot)$ is a Bayesian BBA, i.e., it only has singleton focal elements, it reaches the minimum value 0. When BBA $m(\cdot)$ is a vacuous BBA, i.e., $m(\Theta) = 1$, it reaches the maximum value $\log_2(|\Theta|)$. This definition was proved to have the uniqueness by Ramer [28] and it satisfies all the requirements of the non-specificity measure.

3.3. Measures for total uncertainty in belief functions theory

(1) Aggregated Uncertainty (*AU*) [16]

$$AU(m) = \max\left[-\sum\nolimits_{\theta \in \Theta} p_\theta \log_2 p_\theta \right]$$
$$s.t. \begin{cases} p_\theta \in [0, 1], \forall \theta \in \Theta \\ \sum\limits_{\theta \in \Theta} p_\theta = 1 \\ Bel(A) \leq \sum\limits_{\theta \in A} p_\theta \leq 1 - Bel(\bar{A}),\ \forall A \subseteq \Theta \end{cases} \tag{10}$$

In fact for *AU*, given a BBA, the probability with the maximum Shannon entropy under the constraints established using the given BBA is selected and its corresponding Shannon entropy value is defined as the value for *AU*. Therefore, it is also called as "upper entropy" [29]. It is an aggregated total uncertainty (ATU) measure, which captures both non-specificity and discord. *AU* satisfies all the requirements [29] for uncertainty measure including probability consistency, set consistency, value range, monotonicity, sub-additivity and additivity for the joint BBA in Cartesian space.

(2) Ambiguity Measure (*AM*) [11]

$$AM(m) = -\sum_{\theta \in \Theta} BetP_m(\theta) \log_2 (BetP_m(\theta)) \tag{11}$$

where $BetP_m(\theta) = \sum_{\theta \in B \subseteq \Theta} m(B)/|B|$ is the pignistic probability [30] of a BBA. In fact *AM* uses the Shannon entropy of the pignistic probability of a given BBA to represent the uncertainty.