Knowledge-Based Systems 86 (2015) 288–298

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

A distance measure between intuitionistic fuzzy belief functions

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ARTICLE INFO

Article history:
Received 7 March 2015
Received in revised form 19 June 2015
Accepted 20 June 2015
Available online 25 June 2015

Keywords:
Distance measure
Similarity measure
Belief functions
Intuitionistic fuzzy set

ABSTRACT

Intuitionistic fuzzy (IF) evidence theory, as an extension of Dempster–Shafer theory of evidence to the intuitionistic fuzzy environment, is exploited to process imprecise and vague information. Since its inception, much interest has been concentrated on IF evidence theory. Many works on the belief functions in IF information systems have appeared. However, there is little research on the distance measure between IF belief functions despite the fact that distance measure in classical belief functions has received close attention. In this paper we mainly investigated the distance measure between IF belief functions based on the Euclidean distance between two column vectors. The similarity between focal elements is also taken into account. The distance and similarity measures between IF sets are investigated firstly. A new similarity measure between IF sets along with its properties and proofs is proposed. The positive definiteness of similarity matrix is investigated to guarantee the metric properties of the distance measure. Then a distance measure between IF belief functions is proposed. It is proved that the proposed distance measure is a metric distance. As is illustrated by examples, the distance measure is sensitive to the change of focal elements. Moreover, its applicability for classical belief functions is also demonstrated.

1. Introduction

The Dempster–Shafer theory of evidence, also called belief function theory, is an important method to deal with uncertainty in information systems. Since it was firstly presented by Dempster [1], and was later extended and refined by Shafer [2], the Dempster–Shafer theory, or the D-S theory for short, has generated considerable interest. Its application has extended to many areas such as expert systems [3], fault reasoning [4,5], pattern classification [6–9], information fusion [10], knowledge reduction [11], global positioning system [12], regression analysis [13], and data mining [14].

The theory of fuzzy set, proposed by Zadeh [15], is another mathematical tool for handling uncertainty. It has received a great deal of attention due to its capability in uncertainty reasoning. Therefore, over the last decades, several generalizations of fuzzy set have been introduced in the literature. Intuitionistic fuzzy (IF) set proposed by Atanassov [16,17] is one of the generalizations of fuzzy set which is capable of dealing with vagueness better. A fuzzy set only gives a membership degree to describe an element belonging to a set, while an intuitionistic fuzzy set gives both a membership degree and a non-membership degree. Thus, an IF set is more objective than a fuzzy set to describe the vagueness of information. As a fuzzy set can be reviewed to be a fuzzy event, an IF set is also an IF event.

Relationship between fuzzy set theory and belief function theory has been demonstrated for a long time. Zadeh was the first to generalize the Dempster–Shafer theory to fuzzy sets, based on his work on the concept of information granularity and the theory of possibility [18,19]. He suggested how to compute probabilities of fuzzy events and showed some basic properties of probabilities of fuzzy events [20]. Following Zadeh's work, Ishizuka, Yager, Ogawa, and John Yen have extended the D-S theory to fuzzy sets in di?erent ways [21–24]. As an inception, belief functions on IF events were investigated by Grzegorzewski in [25], where basic properties of probability measures of IF events were studied. Riečan gave an axiomatic characterization of a probability on IF events in [26] and proved a representation theorem for it in [27]. In [28,29], Gerstenkorn and Mańko gave two new definitions of the IF probability: the first probability of an IF set is a real number in [0, 1] using the integral operation, and the second probability of an IF set is also an IF set based on the level sets. In [30], Gerstenkorn and Mańko defined a probability of IF events, which is defined by the membership degree and half of the hesitancy margin of every element, where the probability of an IF set is a real number. Feng et al. [31] proposed a novel pair of belief and

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http://dx.doi.org/10.1016/j.knosys.2015.06.011
0950-7051/

plausibility functions defined by employing intuitionistic fuzzy lower and upper approximation operators.

All the above works on IF belief function theory focused on the determination of the basic probabilities assigned to IF events based on the probabilistic distribution in the universe of discourse. However, the relationship between different IF belief functions has been rarely involved. Since distance and similarity measures between belief functions are crucial to the combination of them, mounts of distance and similarity measures have been proposed in the framework of classical belief function [32]. A comprehensive survey on the distances in classical evidence theory has been carried out by Jousselme and Maupin [32]. They have also pointed out that the distance measures can be further generalized to fuzzy belief functions by substituting measures of similarity between fuzzy sets for the weights $W(A,B)$ [32]. So far, nevertheless, little further investigation on distance between IF belief functions has been presented. The distance measure between IF belief functions is a special distance between two objects. It is also an important tool for analyzing relationship between IF belief functions. Therefore, research on the distance measure between IF belief functions it is desirable.

In this paper, we investigate the distance measure between IF belief functions. We define the distance measure based on the Euclidean distance between two belief function vectors, taking the similarity between IF sets into account. It has been claimed by Bouchard et al. [33] that the positive definiteness of the similarity matrix guaranteed the metric properties of the distance measure. So the similarity matrix must be positive definite to make the distance measure between IF functions be metric. Distance and similarity measures between IF sets are firstly investigated in this paper. In order to define a positive definite similarity matrix, a new similarity measure between IF sets is proposed. Based on the proposed positive definite similarity matrix, a distance measure between IF belief functions is defined. Both proofs and examples are presented to illustrate its properties and performance. For IF belief functions with focal elements consisting of singleton IF set, the concept of basic probability assignment (BPA), belief function and the probability of IF event are identical. So we will adopt the terminology of BPA and belief function for IF events instead of the probability of IF events, which has appeared in other literature.

This paper is structured as follows. Section 2 gives a brief recall of the D-S theory of evidence, IF sets, as well as IF belief function theory. In Section 3, definitions on distance and similarity measures between IF sets are firstly proposed along with their properties. A new similarity measure and its corresponding similarity matrix are defined in Section 3.2. Section 4 presents the distance measure between IF belief functions. Examples and discussions are given in Section 5. In Section 6 we derive the conclusion of this paper.

2. Background

The background material presented in this section deals with the following three main points: (1) the interpretation of Dempster–Shafer theory of evidence, which will be used in this paper to ease the exposition, (2) a brief review of definitions on fuzzy set and IF set, and (3) introduction of intuitionistic fuzzy belief function theory.

2.1. Dempster–Shafer theory of evidence

Dempster–Shafer theory of evidence was modeled based on a finite set of mutually exclusive elements, called the frame of discernment denoted by Ω [1]. The power set of Ω, denoted by 2^{Ω}, contains all possible unions of the sets in Ω including Ω itself. Singleton sets in a frame of discernment Ω will be called atomic sets because they do not contain nonempty subsets. The following definitions are central in the Dempster–Shafer theory.

Definition 2.1. Let $\Omega = \{A_1, A_2, \ldots, A_n\}$ be the frame of discernment. A basic probability assignment (BPA) is a function $m : 2^{\Omega} \to [0,1]$, satisfying the following two conditions:

$$m(\varnothing) = 0 \tag{1}$$

$$\sum_{A \subseteq \Omega} m(A) = 1 \tag{2}$$

where $\varnothing$ denotes empty set, and A is any subset of Ω. For each subset $A \subseteq \Omega$, the value taken by the BPA at A is called the basic probability assigned to A, or the BPA of A for short, denoted by $m(A)$.

Definition 2.2. A subset A of Ω is called the focal element of a belief function m if $m(A) > 0$.

Definition 2.3. For a belief function m on Ω, the belief function and plausibility function which are in one-to-one correspondence with m can be defined respectively as:

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{3}$$

$$Pl(A) = \sum_{B \cap A \neq \varnothing} m(B) = 1 - \sum_{B \cap A = \varnothing} m(B) \tag{4}$$

Definition 2.4 (*[34,35]*). The pignistic transformation maps a belief function m to the so called pignistic probability function. The pignistic transformation of a belief function m on $\Omega = \{A_1, A_2, \ldots, A_n\}$ is given by:

$$BetP(A) = \sum_{B \subseteq \Omega} \frac{|A \cap B|}{|B|} \frac{m(B)}{1 - m(\varnothing)}, \quad \forall A \subseteq \Omega \tag{5}$$

where $|A|$ is the cardinality of set A.

Particularly, when $m(\varnothing) = 0$ and $A \in \Omega$, i.e., A is a singleton set of Ω, we have:

$$BetP(A) = \sum_{A \in B} \frac{m(B)}{|B|}, \quad A = A_1, \ldots, A_n, \; B \subseteq \Omega \tag{6}$$

Definition 2.5. Given two belief functions m_1 and m_2 on Ω, the belief function that results from the application of Dempster's combination rule, denoted as $m_1 \oplus m_2$, or m_{12} for short, is given by:

$$m_1 \oplus m_2(A) = \begin{cases} \frac{\sum_{B \cap C = A} m_1(B) m_2(C)}{1 - \sum_{B \cap C = \varnothing} m_1(B) m_2(C)}, & \forall A \subseteq \Omega, \; A \neq \varnothing \\ 0, \quad A = \varnothing \end{cases} \tag{7}$$

2.2. Intuitionistic fuzzy sets

In this section, we briefly recall the basic concepts related to fuzzy sets and intuitionistic fuzzy sets.

Definition 2.6 [15]. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, then a fuzzy set A in X is defined as:

$$A = \{\langle x, \mu_A(x) \rangle | x \in X\} \tag{8}$$

where $\mu_A(x) : X \to [0,1]$ is the membership degree.