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# Quasi-projective synchronization of fractional-order complex-valued recurrent neural networks

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ABSTRACT

In this paper, without separating the complex-valued neural networks into two real-valued systems, the quasi-projective synchronization of fractional-order complex-valued neural networks is investigated. First, two new fractional-order inequalities are established by using the theory of complex functions, Laplace transform and Mittag-Leffler functions, which generalize traditional inequalities with the first-order derivative in the real domain. Additionally, different from hybrid control schemes given in the previous work concerning the projective synchronization, a simple and linear control strategy is designed in this paper and several criteria are derived to ensure quasi-projective synchronization of the complex-valued neural networks with fractional-order based on the established fractional-order inequalities and the theory of complex functions. Moreover, the error bounds of quasi-projective synchronization are estimated. Especially, some conditions are also presented for the Mittag-Leffler synchronization of the addressed neural networks. Finally, some numerical examples with simulations are provided to show the effectiveness of the derived theoretical results.



## 1. Introduction

As the promotion and generalization of recurrent neural networks in the real domain (Cichocki & Unbehauen, 1993; Ema, Yokoyama, Nakamoto, & Moriizumi, 1989; Matsuoka, Ohoya, & Kawamoto, 1995; Nossek, Gerhard, Tamás, & Chua, 2010; Valle & Castro, 2017; Xing, Jiang, & Hu, 2013), complex-valued recurrent neural networks are established by substituting complex-valued state vectors, connection weights, activation functions, external inputs or outputs for real-valued ones. Currently, complex-valued recurrent neural networks have been received considerable attention due to their comprehensive applications in filtering, computer vision, remote sensing, quantum devices, spatio-temporal analysis of physiological neural devices and systems (Aizenberg, 2017; Zhang, Sui, & Li, 2017). Moreover, the complex-valued recurrent neural networks have much more complicated properties than the real-valued ones which make them possible to solve some problems that cannot be solved by real-valued models such as the detection of symmetry problem and the exclusion OR (XOR) problem. In view of those, more and more researchers recently investigated the dynamic behaviors of complex-value recurrent neural networks and many excellent results were obtained in Aizenberg (2017), Fang and Sun (2014) and Zhang, Sui et al. (2017).

As early as 1695s, the concept of fractional operator (Kilbas, Srivastava, & Trujillo, 2006; Monje, Chen, Vinagre, Xue, & Feliu, 2010; Podlubny, 1999) has been put forward by Leibnitz. For a long time, the theory of fractional calculus is developed mainly as a purely theoretical field of mathematics. However, in the wake of developments in science and technology, many scholars have pointed out that fractional-order calculus is ideally suitable to describe the various materials and processes with memory and hereditary properties (Elfarhani, Jarraya, Abid, & Haddar, 2016), which is not available in integer-order ones. Based on that, it is imperative that the fractional calculus is introduced to complex-valued recurrent neural networks. Lately, more and more researchers discussed fractional-order complex-valued recurrent neural networks and some valuable results were reported on bifurcations (Huang, Cao, Xiao, Alsaedi, & Hayat, 2017; Kaslik & Rădulescu, 2017; Wang, Wang, Li, & Huang, 2017), stability analysis (Rakkiyappan, Velmurugan, & Cao, 2015; Tyagi, Abbas, & Hafayed, 2016; Zhang, Song, & Zhao, 2017), and synchronization (Bao, Park, & Cao, 2016b). It should be noted that these results are mainly focused on stability analysis and there are few results on synchronization control of fractional-order complex-valued neural networks. In addition, the essential technique used in Bao et al. (2016b), Huang, Cao et al. (2017), Kaslik and Rădulescu (2017), Rakkiyappan et al. (2015), Tyagi et al. (2016), Wang, Wang et al. (2017), Wang, Yang et al. (2017) and Zhang, Song et al. (2017) is that the addressed complex-variable systems were first separated into two real-valued systems according to their real and imaginary parts, and then the criteria on

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stability or synchronization were obtained by investigating these real-valued systems. Although the method is effective, the dimensions of the two real-valued systems are double that of the original complex-valued neural networks, and this greatly increases the difficulty of theoretical analysis and the complexity of the derived results. Hence, to overcome or avoid the problem, a natural idea is how to directly discuss the synchronization of the complex-valued neural networks with fractional-order by using the theory of complex functions rather than separating the original complex-valued neural networks into two real-valued systems. Evidently, the problem is valuable and meaningful.

As a typical collective behavior, synchronization has attracted considerable attention due to its theoretical importance and practical applications in various fields such as the cryptography, modeling brain activity, clock synchronization of sensor networks (Annovazzi-Lodi, Donati, & Scire, 1997; Bao, Park, & Cao, 2016a; Chen, Lu, & Zheng, 2015; Chien & Liao, 2005). Up to now, many types of synchronization schemes have been presented such as complete synchronization (Bao, Park, & Cao, 2015; Gu, Yu, & Wang, 2017), anti-synchronization (Huang & Cao, 2017), phase synchronization (Zhang, Wang, & Lin, 2017), and projective synchronization (Song, Song, & Balsera, 2017; Wang, Yang, Hu, & Xu, 2015; Yu, Hu, & Jiang, 2015; Yu, Hu, Jiang, & Fan, 2014; Zhang, Yang, & Wang, 2017; Zheng et al., 2017). Among a great variety of synchronization schemes, projective synchronization, characterized by a scaling factor that the drive system and response system could be synchronized proportionally, is one of the most interesting problems. Particularly, it can be used to extend binary digital to M-nary digital communication for achieving fast communication. Nowadays, projective synchronization has been extended to fractional-order neural networks (Song et al., 2017; Wang et al., 2015; Yu et al., 2015, 2014; Zhang, Yang et al., 2017; Zheng et al., 2017), in which some complex and hybrid controllers including open loop control and switching control were, respectively, designed to investigate the projective synchronization of fractional-order neural networks. Evidently, such complex controllers are inconvenient and undesirable in applications. In addition, their proposed models are real-valued neural networks, and the projective synchronization of complex-valued neural networks with fractional-order has not been considered to the best of our knowledge. Moreover, as pointed out in Huang, Fan, Jia, Wang, and Li (2017) and Yang, Li, Huang, Song, and Chen (2017), in practical synchronization implementations, the synchronization error could not always approach zero with time, but fluctuates within a small bound, that is so-called quasi-synchronization.

Motivated by the above discussions, in this paper, we focus our attention on the quasi-projective synchronization of fractional-order complex-valued recurrent neural networks. The main contribution of this paper lies in the following aspects:

(1) At first, two new inequalities are established under the fractional calculus. The first inequality is related to the Caputo fractional derivative of $(f(t)-h)\overline{(f(t)-h)}$, which extends the corresponding result within the real domain given in Yu et al. (2015), where $f(t), h \in \mathbb{C}$. The second one is an important generalization of the traditional inequality with first-order derivative introduced in Liao (2000) and improves the result given in Wu and Zeng (2017), which plays an essential role in the investigation of quasi-synchronization of fractional-order neural networks.

(2) Second, different from some complex and hybrid controllers given in Song et al. (2017), Wang et al. (2015), Yu et al. (2014), Yu et al. (2015), Zhang, Yang et al. (2017) and Zheng et al. (2017), a linear feedback control scheme is designed to discuss quasi-projective synchronization of complex-valued recurrent neural networks with fractional-order.

(3) Unlike totally the traditional technique in Bao et al. (2016b), Kaslik and Rădulescu (2017), Rakkiyappan et al. (2015), Tyagi et al. (2016), Wang, Yang et al. (2017) and Zhang, Song et al. (2017) and without separating the original complex-valued neural networks into two real-valued systems, several conditions are obtained to ensure quasi-projective synchronization of a class of complex-valued neural networks with fractional-order based on the theory of complex functions. Moreover, the error bound of quasi-projective synchronization is estimated. Especially, the Mittag-Leffler synchronization is also investigated for the addressed neural networks. It is noted that the whole analysis process in this paper is proposed in the complex-valued domain.

The remainder of this paper is organized as follows. In Section 2, some lemmas, definitions and model description are introduced. In Section 3, a linear control scheme is designed and several sufficient conditions for quasi-projective synchronization are obtained. In Section 4, some numerical examples are given to illustrate the feasibility of the proposed method. Finally, some conclusions are drawn in Section 5.

## 2. Preliminaries and model description

In this section, some useful definitions and lemmas are introduced to investigate quasi-projective synchronization of fractional-order complex-valued recurrent neural networks.

**Notations:** Let $\mathbb{C}^n$ be a space composed of all $n$-dimensional complex vectors and $\bar{x}$ be the conjugate of $x \in \mathbb{C}$. $[\mathrm{Re}(x)]$ stands for the integer part of $\mathrm{Re}(x)$ and $\mathrm{Re}(x)$ is the real part of $x$. For a complex number $x = s + ir$, $i = \sqrt{-1}$ is the imaginary unit, $s$ and $r$ are the real and imaginary parts of $x$, respectively. In addition, for any $x_j \in \mathbb{C}$, $|x_j| = \sqrt{x_j \bar{x}_j}$ represents the norm of $x_j$. For any $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$, $\|x\|_2 = (\sum_{j=1}^{n} |x_j|^2)^{\frac{1}{2}}$ represents the norm of $x$. $\mathcal{C}^n([t_0, +\infty), \mathbb{C})$ is a set composed of all continuous and $n$-order differentiable functions from $[t_0, +\infty)$ into $\mathbb{C}$.

### 2.1. Preliminaries

In this paper, Caputo derivative is chosen to deal with fractional-order complex-valued recurrent neural networks.

**Definition 1** (*Kilbas et al., 2006; Monje et al., 2010; Podlubny, 1999 Reimann–Liouville Fractional-order Integral*)**.** Let $\Omega \in [t_0, \infty)$ $(t_0 \geq 0)$ be an interval on the real axis $\mathbb{R}$. Then the Reimann–Liouville fractional-order integral of order $\alpha > 0$ for an integrable function $f(x) : [t_0, +\infty) \to \mathbb{C}$ is defined by

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, $$

here $\Gamma(\alpha)$ is the Gamma function which is defined by

$$ \Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt. $$

**Definition 2** (*Kilbas et al., 2006; Monje et al., 2010; Podlubny, 1999 Caputo Fractional-order Derivative*)**.** The Caputo fractional-order derivative of order $\alpha > 0$ for a function $f(x) \in \mathcal{C}^n([t_0, +\infty), \mathbb{C})$ is defined as

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, $$

where $t \geq t_0$, $n = [\alpha] + 1$ is a positive integer and $[\alpha]$ means the integer part of $\alpha$. Particularly, when $[\alpha] = 0$, that is, $0 < \alpha < 1$,

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(\tau)}{(t-\tau)^{\alpha}} d\tau. $$