Neural Networks 98 (2018) 251–262

Contents lists available at ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Standard representation and unified stability analysis for dynamic artificial neural network models[☆]

Kwang-Ki K. Kim[a,*], Ernesto Ríos Patrón[b], Richard D. Braatz[c]

[a] Department of Electrical Engineering, Inha University, Incheon, Republic of Korea
[b] Petroleum Inst of Mexico, Mexico City, Mexico
[c] Massachusetts Institute of Technology, Cambridge, MA, United States

ARTICLE INFO

Article history:
Received 28 May 2017
Received in revised form 25 September 2017
Accepted 20 November 2017
Available online 2 December 2017

Keywords:
Dynamic artificial neural networks
Nonlinear stability analysis
Absolute stability
Semidefinite programming
Linear matrix inequalities

ABSTRACT

An overview is provided of dynamic artificial neural network models (DANNs) for nonlinear dynamical system identification and control problems, and convex stability conditions are proposed that are less conservative than past results. The three most popular classes of dynamic artificial neural network models are described, with their mathematical representations and architectures followed by transformations based on their block diagrams that are convenient for stability and performance analyses. Classes of nonlinear dynamical systems that are universally approximated by such models are characterized, which include rigorous upper bounds on the approximation errors. A unified framework and linear matrix inequality-based stability conditions are described for different classes of dynamic artificial neural network models that take additional information into account such as local slope restrictions and whether the nonlinearities within the DANNs are odd. A theoretical example shows reduced conservatism obtained by the conditions.

1. Introduction

Black-box identification of nonlinear dynamical systems using artificial neural network (ANN) models have been investigated since the 1980s, with a strong motivation coming from the ability of ANNs to universally approximate static nonlinear functions (Cybenko, 1989; Funahashi, 1989; Hornik, 1989). Cybenko (1989) and Funahashi (1989) proved that an ANN with only one hidden layer can uniformly approximate any continuous function whereas Hornik (1989) studied the universal approximation property of multi-layer ANNs. Subsequent papers showed that the functional range of an ANN is dense for different activation functions (Park & Sandberg, 1991).

This article[1] starts with considering three popular classes of black-box nonlinear dynamical models:

- Neural State-Space Model (NSSM): This state-space model parameterization for a nonlinear dynamical system has nonlinearities parameterized by multilayer feedforward artificial neural networks (FANNs) with one hidden layer;
- Global Input–Output Model (GIOM): This recursive input–output parameterization for a nonlinear dynamical system has nonlinearities parameterized by FANNs;
- Dynamic Recurrent Neural Network (DRNN): This structure is the same as NSSM except with an additional linear recursive term in the state equation.

Stability analysis and controller synthesis based on robust control theory has been extensively studied for NSSMs (Suykens, Moor, & Vandewalle, 1995; Suykens, Vandewalle, & Moor, 1996), which can be rather parsimonious models for some nonlinear dynamical systems. The GIOM allows the future outputs of the dynamical system to be determined purely from a finite number of past observations of the system's measured inputs and outputs (Billings, Jamaluddin, & Chen, 1992; Levin & Narendra, 1995a; Narendra & Parthasarthy, 1990). Since both inputs and outputs to the network are directly observable at each instant of time, static backpropagation or any other supervised training method of system identification can be used to train the network. The application to the adaptive control has been extensively studied (Antsaklis, 1990; Ge & Wang, 2002; Narendra & Mukhopadhyay, 1997). Although similar to NSSMs, DRNNs are more extensively studied in the literature, including

[☆] Preliminary versions of the results in this work were presented in Kim et al. (2011a, 2011b).

* Correspondence to: Hitech Center #629, #100 Inha-ro, Nam-gu, Incheon, Republic of Korea.

E-mail addresses: kwangki.kim@inha.ac.kr (K.-K.K. Kim), erios@www.imp.mx (E.R. Patrón), braatz@mit.edu (R.D. Braatz).

[1] Preliminary versions of the results of this work were presented in conference proceedings (Kim, Patrón, & Braatz, 2011a, 2011b).

https://doi.org/10.1016/j.neunet.2017.11.014
0893-6080/

for large FANNs (Hopfield, 1982; Pineda, 1989). DRNNs have been argued as being well suited for modeling associative memories, and the identification and analysis of DRNNs have been extensively investigated in recent years (Fang & Kincaid, 1996; Grujic & Michel, 1991; Jin, Nikiforuk, & Gupta, 1995; Michel, Farrel, & Porod, 1989).

The ability of the three classes of DANNs to universally approximate nonlinear dynamical systems rely on the universal approximation capability of ANNs for static nonlinear functions. While many results have been reported in the literature, this paper presents all three DANN models in a common representation while filling in the theoretical gaps in the literature, which serves as a capstone to the topic. The common representation is argued to be a useful model structure in its own right, due to its inheritance of all of the universal approximation properties with error bounds derived for the three DANN models.

This paper also considers the stability analysis of DANN models, which is a topic that has been investigated by many researchers (see Michel & Liu, 2002; Suykens et al., 1996 and references cited therein). Stability analysis and the convergence of the state trajectories to equilibria have been studied for DRNNs, with sufficient stability conditions derived using diagonal quadratic Lyapunov functions (Michel et al., 1989) and matrix measures (Fang & Kincaid, 1996). NSSMs have been analyzed by reformulation as NL_q systems for which sufficient conditions for global asymptotic stability (g.a.s.) and input–output stability can be applied (Suykens et al., 1996). However, existing stability conditions are problem-dependent in the sense of being applicable only to specific structures of parameterized models, not to a general representation of DANN models. To construct unified analysis tools, we show that any DANN can be represented in a standard nonlinear operator form (SNOF) and we derive polynomial-time sufficient conditions for the stability of a DANN based on its corresponding SNOF. We also show how existing results in literature can be applied to stability analysis, which are compared to the new stability conditions from a theoretical point of view and with a numerical example.

This paper is organized as follows. Section 2 presents some mathematical notation, definitions, and preliminaries on ANNs and DANNs. Section 3 presents the mathematical descriptions for the three different classes of DANNs with a common block diagram representations to help the reader understand their structures and differences. The approximation properties for each model are given with proofs. Section 4 argues that the common block diagram representation could form the basis for the development of new process identification algorithms and shows the representations of DANNs in terms of the SNOF. Section 5 uses a modified Lur'e–Postnikov function to produce less conservative conditions for g.a.s. for the different classes of SNOFs that represent the DANNs. Section 6 discusses several stability conditions that are applicable to the DANNs. Section 7 concludes the paper.

2. Background

2.1. Mathematical notations and definitions

The notation used in this paper is standard: $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of all nonnegative integers and the set of all nonnegative real numbers, respectively; $\|\cdot\|$ is the Euclidean norm for vectors, or the corresponding induced matrix norm for matrices; 0 and I denote the null matrix whose components are all zeros and the identity matrix of compatible dimension, respectively; the superscript T denotes the transpose of a matrix; ℓ_2^n is the set of all measurable essentially bounded functions from $\mathbb{Z}_+$ to $\mathbb{R}^n$ with ℓ_2^n-norm defined by $\|f\|_{\ell_2^n} \triangleq \sum_{k=0}^{\infty}\|f(k)\| < \infty$ and ℓ_∞^n is the set of all measurable essentially bounded functions from $\mathbb{Z}_+$ to $\mathbb{R}^n$ with ℓ_∞^n-norm defined by $\|f\|_{\ell_\infty^n} \triangleq \max_{1\le i\le n}\left\{\sup_{k\ge 0}|f_i(k)|\right\} < \infty$, where the subscript i denotes the ith element of a vector. For a truncated signal, $f_{[0,\kappa]}(k)$ is defined to have the same value as $f(k)$ at $k \in [0,\kappa]$, $\kappa < \infty$, and is zero for all $k > \kappa$; $\mathcal{C}^p(\mathcal{X})$ denotes the set of p-times continuously differentiable functions on an open set $\mathcal{X}$; $\mathcal{L}^1(\mathcal{X})$ denotes the set of integrable functions on an open set $\mathcal{X}$; $\mathcal{X}^p$ denotes the Cartesian product of $\mathcal{X}$ with itself p times; ι^p is the p-Cartesian product of the interval [0, 1]. The maximum singular value of a matrix $M \in \mathbb{R}^{n\times n}$ is denoted by $\bar{\sigma}(M)$ and the spectral radius of M is denoted by $\rho(M)$. If $M = M^{\mathrm{T}}$ then all the eigenvalues of M are real and $\lambda_{\max}(M)$ denoted the largest eigenvalue. If $M = M^{\mathrm{T}}$ then $M > 0$ and $M < 0$ denote the matrix is positive and negative definite, respectively. Let n and m be positive integers and partition $M \in \mathbb{R}^{(n+m)\times(n+m)}$ as $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. For a matrix $\Delta \in \mathbb{R}^{m\times m}$ such that $\mathrm{I}-D\Delta$ is invertible, define the linear fractional transformation $F_l(M,\Delta) \triangleq A+B\Delta(\mathrm{I}-D\Delta)^{-1}C$. This transformation can be used to define an uncertain autonomous discrete-time system: $x_{k+1} = F_l(M,\Delta)x_k$. Similarly, for $\Omega \in \mathbb{R}^{n\times n}$ such that $\mathrm{I}-A\Omega$ is invertible, define $F_u(M,\Omega) \triangleq D + C\Omega(\mathrm{I}-A\Omega)^{-1}B$. This transformation can be used to define a transfer function matrix, e.g., $G(z) := D + C(z\mathrm{I}-A)^{-1}B = F_u(M,\frac{1}{z}\mathrm{I})$. Define the system norm $\|G\|_\infty \triangleq \max_{0\le\theta\le 2\pi}\bar{\sigma}(G(e^{j\theta}))$ induced by signal 2-norms on the input and output vectors.

The notation of Levin and Narendra (1995b) will be used to describe an ANN: An ANN with only forward connections (called a forward ANN, or FANN) containing L layers of neurons with $(L-2)$ hidden layers, each one with $i_2, i_3, \ldots, i_{L-1}$ neurons respectively, is represented by $\mathcal{N}^{L-1}_{i_1,i_2,\ldots,i_L}$. This network has i_1 inputs and i_L outputs. Therefore, a FANN with 3 inputs, 4 neurons at the hidden layer, and 2 outputs is represented by $\mathcal{N}^2_{3,4,2}$.

This paper considers discrete-time nonlinear dynamical systems of the form $x_{k+1} = f(x_k, u_k, k)$, $y_k = g(x_k, u_k, k)$, where f is locally Lipschitz in all of its arguments, g is continuous in all of its arguments such that the existence of unique solution x is guaranteed, and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$, and $y \in \mathbb{R}^{n_y}$ are the state, control input, and output, respectively. The Lipschitz condition for the state transition map f and the continuity of the output map g are common assumptions used to guarantee the existence of unique solution and assumed to hold for all system equations in this paper, although it might appear as different forms. The subscript k denotes the time instant $k \in \mathbb{Z}_+$.

2.2. Architecture of artificial neural networks

An Artificial Neural Network (ANN) is capable of arbitrarily closely approximating nonlinear functional relationships between bounded input and output variables in the sense that the approximation error can be enforced to be measure zero (Cybenko, 1989; Funahashi, 1989; Hornik, 1989; Park & Sandberg, 1991; Sjöberg, Hjalmarsson, & Ljung, 1994). The basic processing elements of an ANN are referred to as *neurons*, a collection of neurons is referred to as a *layer*, and the collection of interconnected layers forms the ANN. The way in which these layers are connected to each other is known as the *architecture* of the ANN. A neuron, as in Fig. 1a, is a processing element of the form: $y = \gamma\left(\sum_{j=1}^{m} v_j a_j + \beta\right)$ where $\gamma(\cdot)$ represents a nonlinear function known as an *activation function*, $v^{\mathrm{T}} = [v_1, v_2, \ldots, v_m]$ represents a connection parameter vector or weight vector between the neuron and the previous layer, a_j represents the input signals from the previous layer into the neuron, and β represents a bias term. Each neuron is a parameterized mapping $\gamma: \mathbb{R}^m \to \mathbb{R}$. The activation function $\gamma(\cdot)$ is usually chosen to be a monotonic $\mathcal{C}^1$ function bounded in the interval [0, 1] or the interval [−1, 1].

The most common ANN architecture is the Feedforward ANN (FANN) (see Fig. 1b): $y_i = \gamma\left(\sum_{j=1}^{h} W_{ij}\gamma\left(\sum_{p=1}^{m} V_{jp}u_p + \beta_j\right)\right)$ for $i = 1, 2, \ldots, l$, where $V \in \mathbb{R}^{h\times m}$, and V_{jp} represents the weight