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Multistability and instability analysis of recurrent neural networks with time-varying delays

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ABSTRACT

This paper provides new theoretical results on the multistability and instability analysis of recurrent neural networks with time-varying delays. It is shown that such n-neuronal recurrent neural networks have exactly $(4k+3)^{k_0}$ equilibria, $(2k+2)^{k_0}$ of which are locally exponentially stable and the others are unstable, where k_0 is a nonnegative integer such that $k_0 \leq n$. By using the combination method of two different divisions, recurrent neural networks can possess more dynamic properties. This method improves and extends the existing results in the literature. Finally, one numerical example is provided to show the superiority and effectiveness of the presented results.

1. Introduction

Recently, cellular neural networks (CNNs) have attracted much attention from both academic and industry communities due to their wide applications in image processing, pattern recognition, associative memory and their ability to tackle complex problems, for example Chen and Rong (2003), Chen, Zhang, and Lin (2016), Chua and Yang (1988), Cohen and Grossberg (1983), Kosko (1988), Liu, Li, Tong, and Chen (2016c), Liu and Michel (1993), Lu, Wang, and Chen (2011), Maundy and El-Masry (1990), Thiran, Crounse, Chua, and Halser (1995), Wang, Shen, Yin, and Zhang (2015), Wang, Sun, and Mazenc (2016), Wen, Zeng, Chen, and Huang (2017), Wen, Zeng, Huang, Yu, and Xiao (2015) and Yuan and Cao (2005). Recurrent neural networks (RNNs) are regarded as another kind of neural networks, which have more abundant dynamic properties. It is necessary to further research and study recurrent neural network.

The stability analysis of neural networks for steady-state solution (equilibria or invariant orbit) is the prerequisite and foundation in practical application, see, e.g., Cao (2001), Chen, Ge, Wu, and Gong (2015), Chen and Wang (2007), Di Marco, Forti, and Pancioni (2016), Huang, Fan, and Mitra (2017), Nie and Zheng (2015a), Wen, Zeng, Huang, and Zhang (2014), Zeng, Wang, and Liao (2003) and Zhang and Shen (2015). In an associative memory neural network, the dynamic evolution process from any initial state to its adjacent equilibrium points or adjacent periodic orbits can be considered as a process of associative memory, which requires multistability or multiperiodicity to provide theoretical analysis. In other words, in order to have the effect of associative memory in neural networks, memory model is designed for equilibrium points or periodic orbits. In addition, multistability or multiperiodicity is of great interest in both theory and practice (Cao, Feng, & Wang, 2008; Liu, Zeng, & Wang, 2016a; Nie, Zheng, & Cao, 2015; Shayer & Campbell, 2000; Zhang, Yi, & Yu, 2008; Zhang & Zeng, 2016).

In recent years, there are still many interesting topics of the multistability of neural networks and the topics have been widely discussed (Cheng, Lin, & Shih, 2006; Cheng, Lin, Shih, & Tseng, 2015; Di Marco, Forti, & Pancioni, 2017; Kaslik & Sivasundaram, 2011; Liu, Zeng, & Wang, 2016b; Nie, Cao, & Fei, 2013; Nie & Zheng, 2015b; Nie, Zheng, & Cao, 2016; Wang & Chen, 2012, 2014, 2015; Zeng & Wang, 2006; Zhang, Yi, Zhang, & Heng, 2009). It should be noted that most existing results are concerned with the neural networks with bounded activation function or bounded time delays. For instance, in Zeng and Wang (2006), by decomposition of state space $\Re^n$ into 3^n areas, some conditions were derived to ensure the existence of the multiperiodicity of CNNs, and to acquire 2^n stable periodic trajectories. Specially, 3^n equilibria in the Hopfield-type neural networks are obtained in Cheng et al. (2006). Besides, it was shown that convergence and multistability of DM-CNNs in the general case of nonsymmetric interconnections could be investigated in Di Marco et al. (2017).

In order to make storage capacity greater, in Bao and Zeng (2012), the neural networks with discontinuous activation functions were considered, and it was proved that the n-neuronal dynamical networks can obtain $(4k-1)^n$ locally exponentially stable equilibrium points. More generally, in Nie et al. (2013), the

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n-neuronal competitive neural networks with exactly $(2r+1)^n$ equilibria were discussed, $(r+1)^n$ of which were exponentially stable, where the output function of network belonged to a class of piecewise linear functions with $2r(r \geq 1)$ corner points. Moreover, based on the geometrical properties of the activation functions in Gong, Liang, and Zhang (2016), the multistability of complex-valued neural networks with appropriate real-imaginary-type activation functions and distributed delays was addressed. With the development of multistability, the other relevant works could be found in Huang, Zhang, and Wang (2012, 2014), Nie and Zheng (2016) and Wang, Lu, and Chen (2010).

In this paper, by using the combination method of two different divisions, our aim is to further explore the multistability of recurrent neural networks with the piecewise linear activation function. Different from the previous division of state space, we have increased the dimensional division of state space. Some sufficient criteria are obtained to ensure that an n-neuronal recurrent neural network with $(k+1)$-stair activation function can have $(4k+3)^{k_0}$ equilibrium points and $(2k+2)^{k_0}$ of them are locally exponentially stable, where k_0 is a nonnegative integer such that $k_0 \leq n$. By contrast with most of the contributions available in the literature, the dimensional division of state space can lead to neural networks more abundant in dynamic behavior, and some conclusions extend conclusions produced by the division of state space.

Similar activation functions were also presented in Zeng, Huang, and Zheng (2010) and Zeng and Zheng (2012, 2013). The traditional division of state space rely heavily on the dimension of state space. The new way of division(i.e., coupled division) is presented, which reduces dependency on the dimension of state space. Note that by using the dimensional reconstruction and division of state space, the coupled division allows the division of space to be more diverse, and our conclusions extend the existing results of multistability. As a result, it has been well recognized that the different regions of parameter are given by means of coupling division and they can be chosen freely, which is helpful to improve the range of the regions of parameter.

The rest of the paper is organized as follows. Section 2 describes model and preliminaries which will be used later. In Section 3, sufficient conditions are derived for the existence, instability and local stability of the equilibrium points for the recurrent neural networks with time-varying delays. In Section 4, one example is provided to demonstrate the effectiveness of the obtained results. Some concluding remarks are drawn in Section 5.

2. Preliminaries

2.1. Notations

Let $\mathcal{C}([t_0-\tau, t_0], \mathcal{D})$ be the Banach space of functions mapping $[t_0-\tau, t_0]$ into $\mathcal{D} \subseteq \Re^n$ with norm defined $\|\phi\|_\infty = \max_{1\leq i\leq n}\{\sup_{r\in[t_0-\tau,t_0]}|\phi_i(r)|\}$, where $\phi(s) = (\phi_1(s), \phi_2(s), \dots, \phi_n(s))^T \in \mathcal{C}([t_0-\tau, t_0], \mathcal{D})$. Denote $\|x\|_\infty = \max_{1\leq i\leq n}\{|x_i|\}$ as the vector norm of the vector $x = (x_1, x_2, \dots, x_n)^T$; Card$(G)$ as the number of elements in the set G; $|A|$ as the absolute-value matrix of $A = [a_{ij}]$, i.e., $|A| = [|a_{ij}|]$.

For the given integer $k \geq 1$ and the given constant $0 < E_s \in \Re$, $s = 1, 2, \dots, 2k+1$, there exist Z_j^{i-}, $Z_j^{i+} \in \Re, j = 1, 2, \dots, 4k+3$, such that $\forall i \in \{1, 2, \dots, n\}$

$$\begin{aligned} & Z_1^{i-} < Z_1^{i+} < -E_{2k+1} < Z_2^{i-} < Z_2^{i+} \\ & < -E_{2k} < \cdots < -E_1 < Z_{2k+2}^{i-} < Z_{2k+2}^{i+} \\ & < E_1 < Z_{2k+3}^{i-} < Z_{2k+3}^{i+} < E_2 < \cdots < E_{2k} \\ & < Z_{4k+2}^{i-} < Z_{4k+2}^{i+} < E_{2k+1} < Z_{4k+3}^{i-} < Z_{4k+3}^{i+}. \end{aligned}$$

For example, when $k = 1$ and $E_s = 3s-2$, there exist Z_j^{i-}, $Z_j^{i+} \in \Re, j = 1, 2, \dots, 7$, such that $\forall i \in \{1, 2, \dots, n\}$

$$\begin{aligned} & Z_1^{i-} < Z_1^{i+} < -7 < Z_2^{i-} < Z_2^{i+} < -4 < Z_3^{i-} \\ & < Z_3^{i+} < -1 < Z_4^{i-} < Z_4^{i+} < 1 < Z_5^{i-} < Z_5^{i+} \\ & < 4 < Z_6^{i-} < Z_6^{i+} < 7 < Z_7^{i-} < Z_7^{i+}. \end{aligned}$$

Let

$$I_1 = \{i | i = 1, 2, 3, \dots, n\}$$

$$I_2 = \{i | i = 1, 2, 3, \dots, 4k+3\}$$

$$I_3 = \{i \in I_2 | i = 2s-1, s = 1, 2, 3, \dots, 2k+2\}$$

$$I_4 = \{2k+2\}$$

$$I_5 = \{4k+4\}$$

$$\mathcal{D}_{i1} = \{[Z_j^{i-}, Z_j^{i+}] | \forall j \in I_3\}$$

$$\mathcal{D}_{i2} = \{[Z_{2k+2}^{i-}, Z_{2k+2}^{i+}]\}$$

$$\mathcal{D}_{i3} = \{[Z_j^{i-}, Z_j^{i+}] | \forall j \in I_2 - I_3 - I_4\}.$$

Then, $\mathcal{D}_{i1}$ is composed of $(2k+2)$ intervals; $\mathcal{D}_{i2}$ is composed of one interval; $\mathcal{D}_{i3}$ is composed of $2k$ intervals. Since $l_{j(i)}^i \in \mathcal{D}_{i1} \bigcup \mathcal{D}_{i2} \bigcup \mathcal{D}_{i3}$ is an one-dimensional interval, we obtain that for $\forall j(i) \in I_2 \bigcup I_5$

$$\prod_{i=1}^n l_{j(i)}^i = l_{j(1)}^1 \times l_{j(2)}^2 \times \cdots \times l_{j(n)}^n.$$

It is easy to see that any $\prod_{i=1}^n l_{j(i)}^i$ is a compact and convex.

2.2. Model

In this paper, we consider a general class of recurrent neural networks with time-varying delays as follows: $\forall i \in I_1$

$$\begin{aligned} \dot{x}_i(t) = & -x_i(t) + \sum_{j=1}^n a_{ij} f(x_j(t)) \\ & + \sum_{j=1}^n b_{ij} f(x_j(t-\tau_j(t))) + u_i \end{aligned} \qquad (1)$$

where $x = (x_1, x_2, \dots, x_n)^T \in \Re^n$ is the state vector; $A = [a_{ij}]$ and $B = [b_{ij}]$ are connection weight matrices that are not assumed to be symmetric; $u = (u_1, u_2, \dots, u_n)^T \in \Re^n$ is an input vector; for $\forall t \geq t_0$, $\forall j \in \{1, 2, \dots, n\}$, $\tau_j(t)$ with respect to τ satisfies $0 \leq \tau_j(t) \leq \tau = \max_{1\leq i\leq n}\{\sup\{\tau_i(t), t \geq t_0\}\}$. One activation function with $(k+1)$-stair is given by

$$f(x) = \sum_{s=1}^N \frac{(m_{s-1} - m_s)}{2} (|x + E_s| - |x - E_s|) \qquad (2)$$

where $N = 2k+1$, $E_s = 3s-2$, $m_0 = 1$, $m_s = 1 + (-1)^s$, $s = 1, 2, \dots, 2k+1$. In particular, when $k = 0$ in (2), $f(x) = \frac{1}{2}(|x+1| - |x-1|)$ is the common saturated function.

In general, such neural network (1) not only represents the network with delays, but also indicates the network without delays. Denote RNN (1) with activation function (2) to RNN (1′).