Neural Networks 75 (2016) 66–76

Contents lists available at ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Subspace segmentation by dense block and sparse representation

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ARTICLE INFO

Article history:
Received 22 December 2014
Received in revised form 23 November 2015
Accepted 26 November 2015
Available online 7 December 2015

Keywords:
Disjoint
LRR
Subspace segmentation
2-norm

ABSTRACT

Subspace segmentation is a fundamental topic in computer vision and machine learning. However, the success of many popular methods is about independent subspace segmentation instead of the more flexible and realistic disjoint subspace segmentation. Focusing on the disjoint subspaces, we provide theoretical and empirical evidence of inferior performance for popular algorithms such as LRR. To solve these problems, we propose a novel dense block and sparse representation (DBSR) for subspace segmentation and provide related theoretical results. DBSR minimizes a combination of the 1,1-norm and maximum singular value of the representation matrix, leading to a combination of dense block and sparsity. We provide experimental results for synthetic and benchmark data showing that our method can outperform the state-of-the-art.

1. Introduction

Given a set of data points drawn from a union of subspaces, subspace segmentation focuses on partitioning the data points into groups, with each group corresponding to a subspace. It is of substantial interest in broad computer vision and machine learning applications, such as image segmentation (Cheng, Liu, Wang, Huang, & Yan, 2011; Lang, Liu, Yu, & Yan, 2012; Yang, Wright, Ma, & Sastry, 2008), motion segmentation (Rao, Tron, Vidal, & Ma, 2010) and face clustering (Ho, Yang, Lim, Lee, & Kriegman, 2003), and can be formally defined as follows.

Definition 1 (*Subspace Segmentation*)**.** Given a set of data points $X = [x_1, \ldots, x_n]$ drawn from a union of N subspaces $\{S_i\}_{i=1}^k$ with unknown dimensions in a m-dimensional Euclidean space, the task of subspace segmentation is to segment all data points into their respective subspace.

There are mainly two kinds of definitions about the relationship between subspaces discussed in previous work. A collection of N subspaces are independent if and only if $dim(S_1 + \cdots + S_N) = \sum_{i=1}^N dim(S_i)$, where $dim(\cdot)$ denotes the dimensionality of the subspace. Although the literature has often justified certain algorithms by discussing the case of independent subspaces, most of the methods are not only proposed for independent subspace segmentation, as it is clearly overly-restrictive. As an illustration of subspace segmentation, Fig. 1(a) has already appeared in many articles (Elhamifar & Vidal, 2013; Liu, Lin, Yan et al., 2013; Vidal, 2011; Vidal, Ma, & Sastry, 2005), however, it does not satisfy the independence assumption.

An alternative notion focuses on disjoint subspaces (Elhamifar & Vidal, 2010, 2013; Li, Li, Jin, & Xue, 2012; Tang, Liu, Su, & Zhang, 2014). The subspaces $\{S_i\}_{i=1}^N$ are said to be disjoint if and only if every two subspaces only intersect at the origin. If N subspaces are disjoint, we can obtain $dim(S_1+\cdots+S_N) \le \sum_{i=1}^N dim(S_i)$, implying that independence is a special case of disjointness. An illustration of subspace segmentation, used in many papers and shown in Fig. 1(a), contains disjoint but not independent subspaces. In this paper, "disjoint" refers to the disjoint but not independent case for the sake of presentation. In addition, other complex cases of subspace segmentation, such as the overlapping subspaces shown in Fig. 1(b), appear in some applications (Zhang, Cao et al., 2013). The methods (Ma, Yang, Derksen, & Fossum, 2008; Soltanolkotabi & Candès, 2012) can handle this kind of mixture subspaces with nontrivial intersections.

A lot of work (Lu, Feng, Lin, & Yan, 2013; Patel, Nguyen, & Vidal, 2013; Peng, Zhang, & Yi, 2013; Talwalkar, Mackey, Mu, Chang, & Jordan, 2013; Tang et al., 2014; Vidal, 2011; Wang & Xu, 2013; Wang, Xu, & Leng, 2013; Zhang, Sun, He, & Tan, 2013; Zografos, Ellis, & Mester, 2013) has been done on subspace segmentation

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http://dx.doi.org/10.1016/j.neunet.2015.11.011
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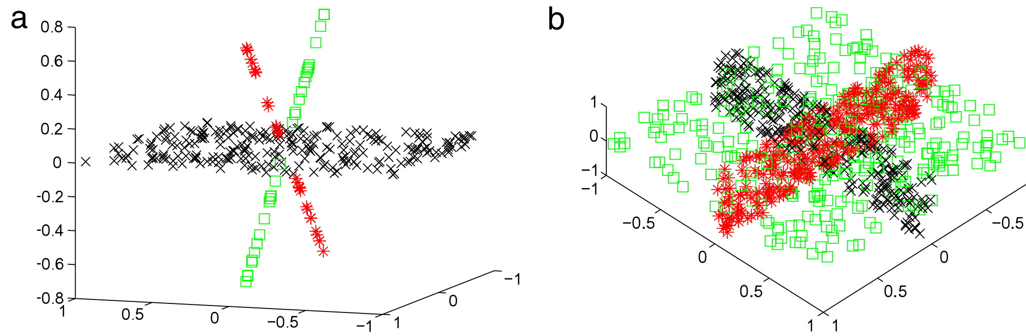

Fig. 1. Two subspace segmentation problems. (a) One plane and two lines; (b) Three planes.

in recent years, and can be roughly divided into four categories: algebraic methods, iterative methods, statistical methods and spectral clustering based methods according to the review (Vidal, 2011). As popular methods, spectral clustering based methods achieve segmentation results by first building the affinity matrix $\mathbf{Z}$, and then applying $(|\mathbf{Z}| + |\mathbf{Z}^T|)/2$ to Normalized Cuts (NCUT) (Shi & Malik, 2000). Because the data are usually assumed to be arranged to satisfy the true segmentation results for the sake of discussion, the block-diagonal affinity matrix is pursued, meaning that data points in the same subspace have the larger weight, while data points in different subspaces are assigned the lower weight. Due to the importance of the block-diagonal property, the paper Feng, Lin, Xu, and Yan (2014) proposed the block-diagonal prior and showed how to integrate it into the model. As two methods generating considerable discussions in the literature, Sparse Subspace Clustering (SSC) (Elhamifar & Vidal, 2009, 2010, 2013) and Low-Rank Representation (LRR) (Liu, Lin, Yan et al., 2013; Liu, Lin, & Yu, 2010) are both spectral clustering based methods. In order that a point can be represented as a linear combination of points in the same subspace, SSC makes a sparsity constraint on the representation matrix while LRR imposes a low-rank restriction.

When the subspaces are independent, LRR and SSC can both achieve a block-diagonal representation matrix. One problem we have observed in practice is that the sparsity constraint in SSC can result in a representation matrix which is so sparse that the weight of many data points in the same subspace is 0, leading to over-segmentation. Theory (Nasihatkon & Hartley, 2011) implies that SSC may over-segment the subspaces for dimensions greater than 3. LRR requires that the sampling is sufficient and the subspaces are independent. The work Liu, Lin, la Torre, and Su (2012) and Liu and Yan (2011) focused on the case of insufficient sampling, proposed Latent Low-Rank Representation (LLRR) and Fix-Rank Representation (FRR), respectively, both of which can also be applied for feature extraction. The work Lu et al. (2012) and Wang, Yuan, Yao, Yan, and Shen (2011) proposed efficient algorithms called Subspace Segmentation via Quadratic Programming (SSQP) and Least Squares Regression (LSR), respectively. When the sampling is sufficient and the subspaces are independent, they can also obtain a block-diagonal solution. Moreover, the work (Lu et al., 2012) estimated the difference of the coefficients in the representation matrix and showed that LSR exhibits strong grouping effect. Putting emphasis on grouping effect and sparsity of the affinity matrix, Lu et al. proposed Correlation Adaptive Subspace Segmentation (CASS) (Lu et al., 2013) for subspace segmentation. The theoretical analysis in Lu et al. (2013) also guaranteed that CASS can estimate a block-diagonal matrix for independent subspace segmentation.

Unfortunately, in real-world problems, the independent subspace assumption is typically violated, but there has been little consideration of optimization methods for disjoint subspace segmentation. The work (Elhamifar & Vidal, 2010, 2013) implies that the normalization preprocessing may help SSC obtain accurate segmentation results when the subspaces are disjoint. The work Tang et al. (2014) proposed Structure-Constrained Low-Rank Representation (SC-LRR) to improve LRR for disjoint subspace segmentation, with theoretical analysis implied that with a predefined weight matrix SC-LRR can achieve block-diagonal affinity matrix. However, it is still difficult to find the weight matrix meeting the requirements of the theorem, even if the provided weight matrix in Tang et al. (2014) can help SC-LRR outperform the state-of-the-art. In this paper, we prove that when subspaces are disjoint but not independent, LRR cannot produce a block-diagonal matrix, and thus may have poor performance. Motivated by our analysis of the characteristics of the affinity matrix obtaining accurate subspace segmentation results in both disjoint and independent cases, we propose a Dense Block and Sparse Representation (DBSR) minimizing the 1,1-norm and 2-norm simultaneously. Compared with SC-LRR, DBSR can perform well without the predefined weight matrix. Moreover, DBSR can incorporate a weight matrix to constrain the structure when it is available. The 2-norm in our model is the first applied for subspace segmentation, with a closed form solution of the 2-norm minimization model provided. Extensive experiments on synthetic and benchmark data demonstrate that DBSR is an effective method. In summary, our contributions are as follows:

1. We present dense block and sparse representation (DBSR) as an effective method for subspace segmentation, whether the subspaces are independent or disjoint.
2. We analyze characteristics of the affinity matrix appropriate for subspace segmentation and confirm that LRR cannot obtain a block-diagonal matrix for disjoint subspaces.
3. Our model is the first to use minimization of 2-norm in this field.

The remainder of this paper is organized as follows. In Section 2, we introduce the main notations. In Section 3 we provide a brief review of LRR. In Section 4, we first analyze the characteristic of the affinity matrix appropriate to subspace segmentation to give our motivation and then propose our model. Numerical solution of our model is provided in Section 5. In Section 6 some extensions about our model have been made. Experimental results are shown in Section 7. Finally, we conclude the paper in Section 8. Proofs are primarily included in an Appendix.

2. Notation

We provide a summary of the main notations in this section. Bold capital symbol denotes a matrix. In particular, $\mathbf{I}$ denotes the identity matrix. The entries of matrices are denoted by bold capital symbol using $[\cdot]$ with subscripts. For example, $[\mathbf{M}]_{ij}$ denotes the (i, j)th entry of the matrix $\mathbf{M}$, $[\mathbf{M}]_{:i}$ denotes the ith column of matrix $\mathbf{M}$, and $[\mathbf{M}]_{i:}$ denotes the ith row of matrix $\mathbf{M}$. Bold capital symbol with superscripts or subscripts still denotes the matrix. For instance, $\mathbf{X}$ denotes a matrix. $\mathbf{X}_i$ also denotes a matrix. Bold lower case symbols denote vectors. The entries of vectors are denoted