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Neural networks letter

A new approach to the stability analysis of continuous-time distributed consensus algorithms[☆]

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ABSTRACT

In this letter, we propose a new approach for the stability analysis of distributed continuous-time consensus algorithms in directed networks with time-dependent communication patterns. Instead of using a continuous-time Lyapunov function, we show how to analyze such a continuous-time algorithm by converting it to a discrete-time model. By using this method, we obtain a more general convergence result than existing ones. An example with numerical simulation is also provided to illustrate the theoretical results.

1. Introduction

In coordination of a network of dynamical agents, a group of agents seeks to agree upon certain quantity of interest. This is the so-called *consensus problem*, which arises in broad areas of applications involving multiagent systems. For example, in formation control of vehicles (Gazi & Passino, 2003; Lin, Broucke, & Francis, 2004), each agent can be a vehicle, and the state variable may represent its position in one-dimensional place; in phase synchronization of oscillators (Jadbabaie, Motee, & Barahona, 2004), each agent is an oscillator, and the state variable is its oscillation phase; in distributed agreement or decision making problem (DeGroot, 1974; Olfati-Saber & Murray, 2003, 2004; Ren & Beard, 2004; Xiao & Boyd, 2004), the state variable can be an abstract decision variable such as the subjective point estimation of the unknown value of some parameter by each individual. Here we just name a few of them. For a review of this area, see the surveys Olfati-Saber (2007), Ren, Beard, and Atkins (2005) and references therein.

If the state variable of each agent converges to a common value as time goes to infinity, then the system reaches a consensus. In order to reach a consensus, usually a distributed consensus protocol is employed. In a distributed consensus protocol, each agent updates its state variable through local interactions, which may be time-dependent and non-bidirectional. Basically, there are two kinds of consensus algorithms: the discrete-time algorithm and continuous-time algorithm. Let $x_i(t) \in \mathbb{R}$ be the state variable of agent i at time t. Then in a discrete-time consensus algorithm, the next state of agent i is a weighted average of its neighbors:

$$x_i(t+1) = \sum_{j\in\mathcal{N}_i} a_{ij}x_j(t), \quad \sum_{j\in\mathcal{N}_i} a_{ij} = 1, \tag{1}$$

while in a continuous-time consensus algorithm, $x_i(t)$ evolves according to a weighted combination of the difference between x_i and its neighbors:

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}[x_j(t) - x_i(t)], \tag{2}$$

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where $a_{ij} > 0$ is the coupling strength from agent j to agent i. And consensus can be achieved if there exists $x^* \in \mathbb{R}$ such that for each i,

$$\lim_{t\to\infty} x_i(t) = x^*.$$

Convergence analysis of the discrete-time consensus algorithm is mainly based on the convergence theory of products of stochastic matrices. For example, see DeGroot (1974), Liu, Lu, and Chen (2011), Lu, Atay, and Jost (2011), Tahbaz-Salehi and Jadbabaie (2008, 2010), and references therein. Meanwhile, continuous-time consensus algorithms have been investigated by various methods. In Olfati-Saber and Murray (2004), the convergence property of continuous-time consensus algorithms has been analyzed based on the eigen-structure of the coupling matrix. However, this approach can only apply to constant couplings but not to time-varying couplings. Continuous-time consensus algorithms with general non-bidirectional and time-dependent couplings are investigated in Moreau (2004). By a Lyapunov function method, the author proved if there exist a fixed time length T and a positive constant δ such that the δ-graph corresponding to the integration of the coupling matrix across each time interval of length T has a spanning tree with a fixed root, then the network will reach a consensus as time tends to infinity. Recently, in Cao, Zheng, and Zhou (2011), the authors proved a necessary and sufficient condition for consensus in undirected continuous-time networks with time-varying connections based on the notion of infinite integral connectivity. Although their result is more general than that in Moreau (2004) concerning undirected networks, it cannot apply to directed networks. And some novel approach should be introduced to extend the result of Moreau (2004) in directed networks.

In this letter, we consider the same continuous-time consensus model as that formulated in Moreau (2004). Instead of using the routine Lyapunov function method, we propose a novel approach, by which the continuous-time consensus algorithm can be converted into a discrete-time one. Then by using the results obtained for the discrete-time consensus algorithms, we can prove a more general convergence result than that in Moreau (2004). More importantly, the new method bridges the gap between the discrete-time and continuous-time consensus algorithms, thus it is a powerful tool for studying time-varying continuous-time systems. Some possible extensions can be expected. For example, it may find applications in the analysis of mixed systems which contains both discrete- and continuous-time parts. Examples of such systems include continuous-time systems with incomplete communications such as sampled-data control (Shen, Wang, & Liu, 2012) or impulsive control. If we can succeed in converting them into discrete-time systems, then we can use some of the results obtained for discrete-time systems such as that in Shen, Wang, and Hung (2010). Furthermore, since consensus can be considered as a special case of synchronization, it is possible to extend this method to the study of synchronization. Although a direct extension to the synchronization analysis of complex networks of general dynamical systems looks quite difficult at this time, it is still possible to extend it on some special systems at first.

The rest of this letter is organized as follows. In Section 2, the mathematical preliminaries are presented; The main results with proof are provided in Section 3; An example with numerical simulation are presented in Section 4; And the paper is concluded in Section 5.

2. Preliminaries

In this section, we provide some definitions and lemmas from matrix theory and graph theory that will be used in this paper.

Let I denote the identity matrix of appropriate dimension, and $\hat{e}$ denote a column vector of appropriate dimension with all entries being 1. For two matrices $A = [a_{ij}]$, $B = [b_{ij}]$ of the same dimension, $A \geq B$ means $a_{ij} \geq b_{ij}$ for each i, j. For a series of matrices $A_1, A_2, \ldots, A_m$, the expression $\prod_{k=1}^{m} A_k = A_m \cdots A_1$ denotes the left product.

Square matrices with nonnegative off-diagonal elements are sometimes referred to as *Metzler* matrices (Luenberger, 1979), and a nonnegative square matrix with each row sum being 1 is called a stochastic matrix.

We also need the following definitions.

Definition 1 (*Liu et al., 2011*)**.** For a nonnegative matrix $A = [a_{ij}]$, and some constant $\delta > 0$, the δ-matrix of A, denoted by $A^\delta = [a_{ij}^\delta]$ is defined as

$$a_{ij}^\delta = \begin{cases} 0, & a_{ij} < \delta \\ a_{ij}, & a_{ij} \geq \delta. \end{cases}$$

Definition 2 (*Wu, 2006*)**.** An $n \times n$ nonnegative matrix $A = [a_{ij}]$ is scrambling, if for each pair of indices (i, j) there exists k ($k = i$ or $k = j$ is allowed) such that both $a_{ik} > 0$ and $a_{jk} > 0$. For some $\delta > 0$, A is δ-scrambling, if the δ-matrix A^δ of A is scrambling.

Example 1. The matrix $\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{bmatrix}$ is scrambling when $a_1, a_2, a_3 > 0$, and is δ-scrambling when $a_1, a_2, a_3 > \delta$. While the matrices $\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & a_1 \\ a_2 & 0 & 0 \\ 0 & a_3 & 0 \end{bmatrix}$ are not scrambling.

Definition 3 (*Wu, 2006*)**.** For a real matrix $A = [a_{ij}]$, the ergodic coefficient, denoted by $\mathrm{EC}(A)$, is defined as

$$\mathrm{EC}(A) = \min_{i,j} \sum_k \min\{a_{ik}, a_{jk}\}.$$

Remark 1. For a stochastic matrix A, $0 \leq \mathrm{EC}(A) \leq 1$, $\mathrm{EC}(A) > 0$ if and only if A is scrambling, and $\mathrm{EC}(A) \geq \delta$ if A is δ-scrambling.

Definition 4 (*Wu, 2006*)**.** For a real matrix $A = [a_{ij}] \in \mathbb{R}^{n\times m}$, its Hajnal diameter, denoted by $\mathrm{HD}(A)$, is defined as:

$$\mathrm{HD}(A) = \max_{i,j} \sum_{k=1}^{m} \max\{0, a_{ik} - a_{jk}\}.$$

Remark 2. For a real matrix A, the Hajnal diameter $\mathrm{HD}(A)$ measures the difference between its rows, and $\mathrm{HD}(A) = 0$ if and only if all the rows of A are identical. Particularly, if A is a stochastic matrix, then its Hajnal diameter $0 \leq \mathrm{HD}(A) \leq 1$. For example, $\mathrm{HD}(A) = 0$ if $A = \hat{e} \cdot v^\top$ where v is a column vector, and $\mathrm{HD}(A) = 1$ if $A = I$.

The following Hajnal's inequality is a basic tool for the convergence analysis of consensus algorithms.

Lemma 1 (*Hajnal, 1958, Paz & Reichaw, 1967*)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a stochastic matrix, then for any real matrix $B \in \mathbb{R}^{n\times m}$,*

$$\mathrm{HD}(AB) \leq [1 - \mathrm{EC}(A)]\mathrm{HD}(B).$$

For example, if $A = I$, then $\mathrm{EC}(A) = 0$ and for any B, $\mathrm{HD}(AB) = \mathrm{HD}(B)$. If $A = \hat{e} \cdot \hat{e}^\top$, then $\mathrm{EC}(A) = 1$ and for any B, $\mathrm{HD}(AB) = 0$. In both cases, we have $\mathrm{HD}(AB) = [1 - \mathrm{EC}(A)]\mathrm{HD}(B)$. If $A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/4 & 3/4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 2/3 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix}$, then $AB = \begin{bmatrix} 1/6 & 11/24 & 3/8 \\ 1/6 & 1/6 & 2/3 \\ 3/8 & 1/2 & 1/8 \end{bmatrix}$, $\mathrm{EC}(A) = 1/4$, $\mathrm{HD}(B) = 3/4$, and $\mathrm{HD}(AB) = 13/24$. Thus, $\mathrm{HD}(AB) < [1 - \mathrm{EC}(A)]\mathrm{HD}(B) = 9/16$.

Remark 3. It should be noted that to use Hajnal's inequality, B need not be a square matrix. Actually, it can be a vector. For