ARTICLE IN PRESS
JID: NEUCOM [m5G;July 30, 2017;10:8]
Neurocomputing 000 (2017) 1–8
Contents lists available at ScienceDirect
Neurocomputing
journal homepage: www.elsevier.com/locate/neucom

A graph-theoretic approach to exponential stability of stochastic complex networks with time-varying delays

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ARTICLE INFO

Article history:
Received 3 December 2016
Revised 27 June 2017
Accepted 11 July 2017
Available online xxx

Communicated by Liu Xiwei

Keywords:
Stochastic complex networks
Time-varying delays
Exponential stability

ABSTRACT

This paper considers a general class of stochastic complex networks with time-varying delays (SCNTVDs). A systematic method of constructing a global Lyapunov function for the complex network is provided by combining graph theory and Lyapunov method. Consequently, some novel and simple sufficient conditions of exponential stability for the SCNTVDs are given. These conditions are presented in terms of the topological structure of networks. In addition, to illustrate the effectiveness and applicability of the proposed theory, a practical model in physics is studied and the numerical simulation is also given.

1. Introduction

During the past decades, study on complex networks has been an active area due to their applications in the fields such as neural networks, telephone call graphs, internet and social networks [1–5]. Most applications focus on the system's dynamic properties, especially for the stability. Many results have been presented in recent years [6–12].

In the implementation of practical complex networks, time delays are ubiquitous not only in the subsystems but also in the process of transmission, due to the finite switching speed of the amplifiers and communication time. The delays are usually not identical and time-varying. The existence of time delays is frequently a source of undesirable dynamic behaviors, e.g. oscillation and instability. In addition to delays, complex networks are often perturbed by various types of stochastic environmental "noise" [13–16]. It has been known that a system could be stabilized or destabilized by certain stochastic input. Thus, it is more realistic to incorporate time-varying delays and random fluctuations in complex networks [17,18].

In this paper, a class of general stochastic complex networks with time-varying delays (SCNTVDs) is considered. The model contains some previous models as special cases. Here we only mention [7–9,17–20]. However, it is very complicated to study the dynamic properties of SCNTVDs, and these complexities are mainly caused by the nonlinearity, random nature, time delays and topological structure of the networks. Particularly, as shown in Example 1, it is challenging to analyze the effects of topological structure on the dynamic properties of complex networks.

Example 1. Consider a complex network, i.e., the following two-dimensional system

$$\begin{cases} x_1'(t) = -ax_1(t) + \alpha_{21}x_2(t), \\ x_2'(t) = bx_2(t) - cx_2^2(t) + \alpha_{12}x_1(t), \end{cases} \tag{1}$$

where a, b, c are positive constants, with initial value $x_1(0) > 0$, $x_2(0) > 0$ and a coupled matrix

$$A = \begin{pmatrix} 0 & \alpha_{21} \\ \alpha_{12} & 0 \end{pmatrix}.$$

System (1) can be regarded as the coupling result of two systems:

$$\text{Marthus system}: \; x_1'(t) = -ax_1(t) \tag{2}$$

and

$$\text{Logistic system}: \; x_2'(t) = bx_2(t) - cx_2^2(t), \tag{3}$$

via coupled matrix A. It is easy to show that system (2) has an asymptotically stable equilibrium $x_1^* = 0$. System (3) has an equilibrium $x_2^* = b/c$ and for $x_2(0) > 0$, it follows that

$$\lim_{t\to\infty} x_2(t) = \frac{b}{c}.$$

However, if we choose different coupling matrix, the stability result of coupled system (1) may be absolutely different. For example, if

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http://dx.doi.org/10.1016/j.neucom.2017.07.026
0925-2312/

Please cite this article as: J. Feng, C. Xu, A graph-theoretic approach to exponential stability of stochastic complex networks with time-varying delays, Neurocomputing (2017), http://dx.doi.org/10.1016/j.neucom.2017.07.026

$\alpha_{12} = 0, \alpha_{21} = 1$, then coupled system (1) has an unstable equilibrium (0, b/c); While if $\alpha_{12} = 1, \alpha_{21} = 0$, then the equilibrium (0, b/c) is asymptotically stable.

Based on the above example and the construction for stable complex networks, the first topic of this paper is

P1: What is the relation between the stability of SCNTVDs and the topological structure of the networks?

Moreover, it is well known that Lyapunov method plays an important role in the study of system's stability. But it is quite not easy to construct an appropriate Lyapunov function for a special complex network and this is a well-known disadvantage of Lyapunov method. However, in practice, to make a progress, different fields have suppressed certain complications for complex networks. For example, in nonlinear dynamics the simple and nearly identical dynamical systems are coupled together in simple, regular ways [1]. These simplifications make that any issues of structural complexity are avoided and the system's potentially formidable dynamics could be studied intensively. Hence, in many fields, the subsystems of complex networks are stable and their Lyapunov functions are known or can be constructed without difficulty. So the second topic focuses on

P2: If the Lyapunov function of each individual subsystem is known, then how to construct a stable complex network and its Lyapunov function will be a central problem to cope with.

Recently, many scholars try to combine the Lyapunov method with other approaches, such as linear matrix inequality [21] and graph-theoretic method [7,9], to give the stability conditions. In [7,9], Li et al. explored the global stability for general coupled systems of ordinary differential equations on networks by graph-theoretic approach. Motivated by this work, some novel sufficient conditions of global asymptotical stability for some mathematical models were presented in [10,19,22–24]. In [8,25], the networks with constant delay have been investigated by the graph-theoretic method. Moreover, it has been revealed that this approach also works well for some stochastic cases without delay [20,26] and for some discrete-time complex networks [27,28]. In [18], global exponential stability for stochastic networks of coupled oscillators with variable delays has been reported by the graph-theoretic approach.

The purpose of the paper is to continue the work of [18,20,25] for a general class of SCNTVDs. More specifically, a theoretical framework for constructing a global Lyapunov function of SCNTVDs will be built, and further, some general and practically plausible global exponential stability criteria for SCNTVDs will be rigorously established. A key new feature in our results is that these criteria have a close relation to the topological structure of the networks.

The remainder of this paper is organized as follows. In Section 2, some basic definitions, lemma and model formulations are given. In Section 3, the theoretical results are presented. Some novel sufficient conditions ensuring pth moment exponential stability (ME-stability) and almost surely exponential stability (ASE-stability) for the model are given. To illustrate the correctness and the effectiveness of the theoretical results, the main results are applied directly to stochastic coupled oscillators on a network in Section 4. Finally, the conclusions are drawn in Section 5.

2. Preliminaries

Some useful notations and a useful lemma in graph theory are stated in Section 2.1. Then the model formulations are presented in Section 2.2.

2.1. Mathematical preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions [15], and $W(\cdot)$ be a one-dimensional Brownian motion defined on the space. The mathematical expectation with respect to the given probability measure $\mathbb{P}$ is denoted by $\mathbb{E}(\cdot)$. Write $|\cdot|$ for the Euclidean norm for vectors or the trace norm for matrices. Denote by $C([-\tau, 0]; \mathbb{R}^n)$ the space of continuous functions $x : [-\tau, 0] \to \mathbb{R}^n$ with norm $\|x\| = \sup_{-\tau\leq u\leq 0} |x(u)|$. $L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ stands for the family of $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables y such that $\mathbb{E}\|y\|^2 < \infty$. The notations $\mathbb{Z}^+ = \{1, 2, \ldots\}$, $\mathbb{L} = \{1, 2, \ldots, l\}$, $\mathbb{R}^1_+ = [0, +\infty)$ and $m = \sum_{i=1}^l m_i$ for $m_i \in \mathbb{Z}^+$ are used. And then we write $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^1_+; \mathbb{R}^1_+)$ for the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^n \times \mathbb{R}^1_+$ that are continuously twice differentiable in x and once in t.

Then we provide some essential concepts of graph theory.

A digraph $\mathcal{G} = (\mathbb{L}, E)$ contains a set of vertices $\mathbb{L}$ and a set E of arcs (k, s) leading from initial vertex k to terminal vertex s, satisfying $(k, k) \notin E$ for all $k \in \mathbb{L}$. A digraph $\mathcal{G}$ is weighted if each arc (s, k) is assigned a positive weight a_{ks}. $A = (a_{ks})_{l\times l}$ is meant the weight matrix of $\mathcal{G}$ where $a_{ks} > 0$ if and only if there exists an arc (k, s) in $\mathcal{G}$. Denote the Laplacian matrix of $\mathcal{G}$ by $L(\mathcal{G}) = (\epsilon_{ks})_{l\times l}$ where

$$\epsilon_{ks} = \begin{cases} \sum_{r\neq k} a_{kr}, & k = s, \\ -a_{ks}, & k \neq s. \end{cases}$$

The following lemma in [7] will be used in the proof of our main results.

Lemma 1. *Let $l \geq 2$ and c_i denote the cofactor of the ith diagonal element of $L(\mathcal{G})$. Then the following identity holds:*

$$\sum_{i,j=1}^{l} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r)\in E(C_{\mathcal{Q}})} F_{rs}(x_r, x_s).$$

Here for any $i, j \in \mathbb{L}$, $F_{ij}(x_i, x_j)$ is arbitrary function, $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $C_{\mathcal{Q}}$ denotes the directed cycle of $\mathcal{Q}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_i > 0$ for $i \in \mathbb{L}$.

2.2. Model formulations

To demonstrate the construction of a general SCNTVD on a digraph $\mathcal{G}$, we first divide the network into $l \geq 2$ individual subsystems, and then connect them together according to the digraph $\mathcal{G}$.

We first assume that the dynamics of the kth ($k \in \mathbb{L}$) subsystem is described by

$$\begin{aligned} dx_k(t) = &f_k(x_k(t), x_k(t-\tau_k(t)), t)dt \\ &+ g_k(x_k(t), x_k(t-\tau_k(t)), t)dW(t), \quad t \geq 0, \end{aligned} \tag{4}$$

where $x_k(t) \in \mathbb{R}^{m_k}$ represents the state of the kth subsystem, the functions of general form $f_k, g_k : \mathbb{R}^{m_k} \times \mathbb{R}^{m_k} \times \mathbb{R}^1_+ \to \mathbb{R}^{m_k}$ are referred as drift coefficient and diffusion coefficient, respectively, and $\tau_k(t) : \mathbb{R}^1_+ \to [0, \tau]$ stands for the time delay existing in the kth subsystem, in which τ is a constant.

Secondly, for any given $k, h \in \mathbb{L}$, assume that the connection influences of the hth subsystem on the kth subsystem for the drift coefficient f_k and the diffusion coefficient g_k are described by functions $H_{kh}(x_h(t-\tau^{(kh)}(t)), t)$ and $N_{kh}(x_h(t-\tau^{(kh)}(t)), t)$, respectively. Here $H_{kk} = N_{kk} = 0$ and $H_{kh} = N_{kh} = 0$ if and only if there exists no influence from the h-th subsystem to the kth subsystem, and $\tau^{(kh)} : \mathbb{R}^1_+ \to [0, \tau]$ reflects the transmissible time from h-subsystem to k-subsystem.

Hence, by incorporating the connection effect of each other, we replace f_k and g_k by $f_k + \sum_{h=1}^l H_{kh}$ and $g_k + \sum_{h=1}^l N_{kh}$, respectively. Thus an m-dimensional ($m = \sum_{k=1}^l m_k$) SCNTVD can be described as follows: