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Continuous attractors of higher-order recurrent neural networks with infinite neurons

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ABSTRACT

This paper investigates continuous attractors of higher-order recurrent neural networks (RNN) with infinite neurons (HRNNwIN). By employing the linearization technique, we present some new criteria on stable, semi-stable, positive semi-global stable and unstable continuous attractors. We also study the continuous attractors of this network under Lognormal distribution except Gaussian distribution. Finally, some simulations are finally carried out to further illustrate the developed theory.

1. Introduction

Continuous attractors, as an important mathematical tool, are used to represent the coding information of active neurons in the brain given external stimuli, such as orientation, moving direction and spatial location of objects, or other continuous features [1–7]. In the brain a neural system acquires its network structure to be recurrent connection. To elucidate brain functions and track the dynamics of neural systems, thus, the core is to understand the dynamics (continuous attractors) of recurrent neural networks (RNN).

For continuous attractors, there are two classes of RNN: (i) RNN with finite neurons [8,9] and (ii) RNN with infinite neurons (RNNwIN). Because it is straightforward that the (ii) model returns to the (i) model, the (ii) model is adopted by many researchers. In this paper, we study continuous attractors of higher-order RNN with infinite neurons (HRNNwIN) under Gaussian distribution (GD) and Lognormal distribution (LD). There are three motivations to explain why we do this work.

Firstly, many works on continuous attractor mainly ground on GD, as the distributions of firing rates and synaptic strengths of neurons are well described by the GD. And the paper [10] analyzes the effects of rate variations by solving a Poisson process. However, in the primary auditory cortex, the distributions of firing rates can be described by a LD [11–13]. So, it is very necessary to study continuous attractors with LD for adapting to the new discovering distributions of firing rates in neural systems.

Secondly, RNNwIN is dissolved into a simple differential equation [1–4,7] for studying the stable mono-continuous attractor of this network. Using the linear continuous attractor caused by a free parameter, the network can only track an activity pattern. But in brain, an external stimuli results in multiple activity patterns of the neural system. For example, fMRI response of MT neurons could have activated multiple patterns with a stimulus [14]. Luckily, higher-order interactions can obtain multistability of networks. Thus, second-order interactions are added into RNNwIN so that we can obtain more continuous attractors to track multiple activity patterns given a stimuli.

Thirdly, RNN with stable continuous attractors is also applied to information retrieval [17] and pattern association [18]. In information retrieval, the stronger the memory capacity is, the stronger the retrieval capability is. But a continuous attractor of RNN can store a memory of eye position with analog neural encoding [15,16]. Exactly, many authors [19–27] also study the dynamics of high-order neural networks due to the fact that they have a greater storage capacity than lower-order neural networks. So, higher-order interactions have the power to provide more continuous attractors for the applications of associative memory storage and information retrieval.

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To the best of our knowledge, there are few results on continuous attractors of HRNNwIN. In this paper, we add the higher-order connections to the RNNwIN so that the networks can obtain the strong storage capacity and track multiple activity patterns. We give a general integral condition so that HRNNwIN can be dissolved into a one-order polynomial differential equation and we also study the continuous attractors of HRNNwIN under GD and LD.

The reminder of the paper is organized as follows. Preliminaries are given in Section 2. The continuous attractors of the HRNNwIN are analyzed in Section 3. Some simulations are carried out to further verify the developed theory in Section 4. The final section offers the conclusion of this paper.

2. Preliminaries

Let us consider a continuous stimulus ξ encoded by an M neurons. For convenience, we assume $M\to\infty$ and consider a continuous neural field model. $X(\xi,t)$ is denoted by the internal state of neurons whose preferred stimulus ξ and $r(\eta,t)$ is denoted by the firing rate of neurons. The dynamics of the HRNNwIN is described by the following dynamic equation:

$$\dot{X}(\xi,t)=-X(\xi,t)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(\xi,\eta,\varphi)\cdot r(\eta,t)\cdot r(\varphi,t)\,d\eta\,d\varphi+I(\xi) \quad (1)$$

$$r(\eta,t)=\frac{X(\eta,t)^2}{1+\rho\int_{-\infty}^{\infty}X(\eta',t)^2\,d\eta'} \quad (2)$$

for $t\geq 0$, where $\xi\in\mathbf{R}$; ρ is a positive constant; $\dot{X}(\xi,t)$ denotes the first derivatives of $X(\xi,t)$ with respect to t; $W(\xi,\eta,\varphi)$ denotes the higher-order connection weights of the neural networks; $I(\xi)$ denotes the external stimuli to neuron ξ.

The trajectory of the network (1) is represented as

$$X(\xi,t)=x(\mu,t)\cdot f(\xi,\mu)$$

where $\mu\in\mathbf{R}$ is a free parameter corresponded to the memory stored in the network and $f(\xi,\mu)$ is a bounded, nonnegative and continuous function. The weight matrixes are denoted by

$$W(\xi,\eta,\varphi)=w\cdot g(\xi,\eta,\varphi)$$

where w is a constant and $g(\xi,\eta,\varphi)$ are bounded, nonnegative and continuous function. And the external stimuli is denoted by

$$I(\xi)=K(\mu)\cdot f(\xi,\mu)$$

where $K(\mu)$ is a bounded, nonnegative and continuous function.

Lemma 1. *If the generalized integrals*

$$\int_{-\infty}^{\infty}f(\xi,\mu)^2\,d\xi=c \quad (3)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi,\eta,\varphi)\cdot f(\eta,\mu)^2\cdot f(\varphi,\mu)^2\,d\eta\,d\varphi=r\cdot f(\xi,\mu) \quad (4)$$

then the HRNNwIN (1) can be rewritten as

$$\dot{x}(\mu,t)=-a_5x(\mu,t)^5+a_4x(\mu,t)^4-a_3x(\mu,t)^3-x(\mu,t)+K(\mu) \quad (5)$$

where r and c are positive constants, $a_5=1/\hat{x}\cdot(c\cdot\rho)^2$, $a_4=1/\hat{x}\cdot w\cdot r$, $a_3=2/\hat{x}\cdot c\cdot\rho$ and $\hat{x}=(1+c\rho x(\mu,t)^2)^2$.

We denote that $|\cdot|$ is the absolute value of a real number and give the following definitions.

Definition 1. A set $X^*(\xi)=\{x^*(\mu)\cdot f(\xi,\mu)|\xi,\mu\in\mathbf{R}\}$ is called an equilibrium point of (1), if it satisfies

$$-X^*(\xi)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(\xi,\eta,\varphi)\cdot r^*(\eta)\cdot r^*(\varphi)\,d\eta\,d\varphi+I(\xi)=0$$

$$r^*(\eta)=\frac{X^*(\eta)^2}{1+\rho\int_{-\infty}^{\infty}X^*(\eta')^2\,d\eta'}$$

Definition 2. An equilibrium $\{x^*(\mu)\cdot f(\xi,\mu)|\xi,\mu\in\mathbf{R}\}$ of (1) is said to be stable (positive semi-stable (PSS), or negative semi-stable (NSS)) if given any $\varepsilon>0$, there exists a $\delta>0$ such that $|x(\mu,0)-x^*(\mu)|<\delta$ (or $x(\mu,0)-x^*(\mu)<\delta$, or $x^*(\mu)-x(\mu,0)<\delta$) implies that $|x(\mu,t)-x^*(\mu)|<\varepsilon$ (or $x(\mu,t)-x^*(\mu)<\varepsilon$, or $x^*(\mu)-x(\mu,t)<\varepsilon$) for all $t\geq 0$. And a real equilibrium is called unstable if it is not stable.

Definition 3. An equilibrium $\{x^*(\mu)\cdot f(\xi,\mu)|\xi,\mu\in\mathbf{R}\}$ of (1) is said to be positive semi-global stable (PSGS) if it is a PSS equilibrium and there exists a $b\in\mathbf{R}$ such that $\{x(\mu,0)>b|\mu\in\mathbf{R}\}$ implies that

$$\lim_{t\to+\infty}x(\mu,t)=x^*(\mu)$$

Definition 4. (Yu et al. [3]). A set of equilibria is said to be a continuous attractor of (1) if it is a connected set.

Definition 5. A set of equilibria C is said to be a stable (PSS, NSS, or unstable) continuous attractor of (1) if C is a continuous attractor and each equilibrium of C is stable (PSS, NSS, or unstable). A set of equilibria C is said to be a semi-stable continuous attractor of (1) if C is a continuous attractor and each equilibrium of C is stable or PSS, or NSS.

Definition 6. A set of equilibria C is said to be a PSGS continuous attractor of (1) if C is a continuous attractor and there exists a PSGS equilibrium of C.

Lemma 2 (*www.webskate101.com [28]*). *Let (X,α) and (Y,β) be metric spaces and suppose that $f:X\longrightarrow Y$ is a continuous function. If $A\subseteq X$ is connected, then f(A) is a connected in Y.*

Lemma 3. *Suppose that*

$$h(x)=-a_5x^5+a_4x^4-a_3x^3-x+K(\mu)$$

The 5-degree polynomial $h(x)$ is decomposed into

$$h(x)=v(x-\lambda_1(\mu))^{m_1}\cdots(x-\lambda_j(\mu))^{m_j}\cdots(x-\lambda_l(\mu))^{m_l}\cdot(x^2-p_1(\mu)x+q_1(\mu))^{k_1}\cdots(x^2-p_i(\mu)x+q_i(\mu))^{k_i}\cdots(x^2-p_s(\mu)x+q_s(\mu))^{k_s} \quad (6)$$

where $v=-a_5(v<0)$ is the leading coefficient, $\lambda_j\ (\mu)(1\leq j\leq l)$ are real roots, $m_j\geq 1$ is the multiplicity of the root $\lambda_j(\mu)$, $\lambda_j(\mu)$ depends continuously on the coefficient μ, $p_i^2(\mu)-4q_i(\mu)<0\ (1\leq i\leq s)$ and $\sum_{j=1}^{l}m_j+2\sum_{i=1}^{s}k_i=5$..

Throughout this paper, the real root set of $-a_5x^5+a_4x^4-a_3x^3-x+K(\mu)=0$ is denoted by $L=\{\lambda_j(\mu)|1\leq j\leq l\}$ and we denote that

$$H(\lambda_j(\mu))=v(x-\lambda_1(\mu))^{m_1}\cdots(x-\lambda_{j-1}(\mu))^{m_{j-1}}\cdot(x-\lambda_{j+1}(\mu))^{m_{j+1}}\cdots(x-\lambda_l(\mu))^{m_l}\cdot(x^2-p_1(\mu)x+q_1(\mu))^{k_1}\cdots(x^2-p_i(\mu)x+q_i(\mu))^{k_i}\cdots(x^2-p_s(\mu)x+q_s(\mu))^{k_s},\quad 1\leq j\leq l,\quad 1\leq i\leq s \quad (7)$$

3. Continuous attractors of HRNNwIN

In this section, we will study the stable, unstable and semi-global stable nonlinear continuous attractors of the HRNNwIN under GD and LD, which are, respectively, described by

$$y(x)=\frac{1}{\delta\sqrt{2\pi}}\cdot\exp\left(-\frac{(x-\mu)^2}{2\delta^2}\right),\quad x\in\mathbf{R}$$

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