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Holes in 2-convex point sets ☆

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ABSTRACT

Let S be a set of n points in the plane in general position (no three points from S are collinear). For a positive integer k, a k-*hole* in S is a convex polygon with k vertices from S and no points of S in its interior. For a positive integer l, a simple polygon P is l-*convex* if no straight line intersects the interior of P in more than l connected components. A point set S is l-*convex* if there exists an l-convex polygonization of S.
Considering a typical Erdős–Szekeres-type problem, we show that every 2-convex point set of size n contains an $\Omega(\log n)$-hole. In comparison, it is well known that there exist arbitrarily large point sets in general position with no 7-hole. Further, we show that our bound is tight by constructing 2-convex point sets in which every hole has size $O(\log n)$.

1. Introduction

Let S be a set of n points in the plane in *general position*, that is, the set S does not contain a collinear point triple. Throughout the whole paper we only consider point sets that are finite and in general position. A convex polygon H is a *hole* in S if its vertices are points of S and the interior of H contains no points of S. If a hole H of S has k vertices, then we say that H is a k-*hole* in S and k is the *size* of the hole. In addition, we regard single points of S as *1-holes* in S and segments determined by two points from S as *2-holes* in S. Slightly abusing the notation, we sometimes use the terms "hole" and "k-hole" also for the set of vertices of a hole and a k-hole, respectively. We remark that in some papers the definition of a hole H allows H to be non-convex.

Erdős [6] asked for the smallest integer $h(k)$ such that every set of $h(k)$ points in general position in the plane contains at least one k-hole. It is easy to check that $h(4) = 5$ and Harborth [8] showed $h(5) = 10$. After this result, the question

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of Erdős was settled in two phases: first, Horton showed that there are arbitrarily large point sets without 7-holes [10]. Around 25 years later, Gerken [7] and Nicolás [11] independently showed that sets with enough points always contain a 6-hole. A recent summary of known results, together with some bounds on the minimum number of k-holes in a set of n points, can be found in [3]. In the current paper, we consider this question for a restricted class of point sets.

The notion of convexity is central in discrete geometry and it has been generalized in a number of ways. Convex polygons can be characterized by looking at their intersections with straight lines: A simple polygon P is convex if and only if $P \cap \ell$ is connected for every line ℓ. Aichholzer et al. [1] extended this property to *l-convex polygons:* For a positive integer l, a simple polygon P is *l-convex* if there exists no straight line that intersects the interior of P in more than l connected components. Clearly, a convex polygon is 1-convex. An extensive study of l-convex polygons can be found in [1].

Let P be an l-convex polygon. We use ∂P to denote the boundary of P. It follows from the definition of l-convexity that every line that does not contain an edge of P intersects ∂P in at most $2l$ points. In particular, every line that does not contain an edge of a 2-convex polygon intersects the boundary of this polygon in at most four points.

In [2], the notion of l-convexity was transcribed to finite point sets. A point set S is *l-convex* if there exists a polygonization $P(S)$ of S such that $P(S)$ is an l-convex polygon. Here, a *polygonization* of S is a closed tour on S whose straight-line embedding in the plane determines a simple polygon. Note that an l-convex polygon or point set is also $(l+1)$-convex.

The problem of deciding whether a set of n points is 2-convex can be solved in polynomial time with respect to n. Aichholzer et al. [2] provided an algorithm that solves this problem in time $O(n^{13})$. They also showed that the problem of deciding whether a point set is 3-convex is NP-complete.

In this paper, we consider the following Erdős–Szekeres-type problem for 2-convex point sets: What is the smallest number $f(k)$ such that any 2-convex point set of size $f(k)$ contains a k-hole?

We show that every 2-convex point set of size n contains a hole of size $\Omega(\log n)$, implying that $f(k)$ is finite for any $k > 0$ (Section 3). Our proof actually yields an algorithm that, given a 2-convex set of n points, finds a hole of size $\Omega(\log n)$ in polynomial time with respect to n. Further, we show that our bound is tight by providing, for every integer n, a construction of 2-convex point sets of size n in which all holes have size $O(\log n)$ (Section 4). It is natural in this context to wonder about the convexity of large sets that only contain holes of constant size. We provide an asymptotically tight lower bound $\Omega(\sqrt{n})$ on the convexity of so-called Horton sets of size n (Section 5).

This problem has also been considered for a different definition of l-convexity. Valtr [14] defined l-convex point sets as sets where no triangle determined by three points of the set contains more than l points in its interior. Note that with this notion of convexity, convex polygons are 0-convex. Valtr [14] showed that there exists a constant $N(l,k)$ such that every l-convex set with at least $N(l,k)$ points contains a k-hole. The exact values of $N(1,k)$ are given in [16]. Consult [5,15] for the best known bounds on $N(l,k)$ for $l \geq 2$. Yet another definition of convexity of a point set is discussed by Arkin et al. [4], who considered the minimum number of reflexive points any polygonization of a given point set might have.

Although some statements in the paper seem intuitively clear, despite our efforts, the presented rigorous proofs are quite technical. One reason for this is the necessity to take into account also singular cases when a line or a ray shares a whole segment with ∂P or if it touches ∂P in some point of S.

2. Properties of 2-convex polygons

The proof of our main result is based on the structure of 2-convex polygons shown in [1]. Let P be a simple polygon and let $\mathrm{CH}(P)$ be its convex hull. In the following, we denote a connected piecewise linear simple arc as a *chain*. We denote with $\langle p_i, \ldots, p_j \rangle$ the chain that starts at p_i and ends at p_j and that traces along ∂P in counterclockwise order. The points p_i and p_j are called the *endpoints* of $\langle p_i, \ldots, p_j \rangle$.

A *pocket* K of P is a chain $\langle p_0, \ldots, p_t \rangle$ such that its two endpoints p_0 and p_t are its only vertices of $\mathrm{CH}(P)$. The segment $p_0 p_t$ is called the *lid* of K. If a pocket consists solely of a single convex hull edge of P, we call it a *trivial pocket*.

The following three observations follow directly from the definitions. First, the lids of the pockets of P form the boundary of $\mathrm{CH}(P)$. Second, every vertex of $\mathrm{CH}(P)$ lies in exactly two pockets. And finally, if a line intersects the interior of $\mathrm{CH}(P)$ then its intersections with ∂P cannot all lie in a single pocket.

The structure of non-trivial pockets is quite simple and it will be crucial in our proof. It is outlined in the following two lemmas.

Lemma 1 *([1, Lemma 12]). Let $K = \langle p_0, \ldots, p_t \rangle$ be a non-trivial pocket of a 2-convex polygon between two extreme points p_0 and p_t. Then there are integers r and s with $0 \leq r < s < t$ such that K consists of three chains $C_1 = \langle p_0, \ldots, p_r \rangle$, $C_2 = \langle p_{r+1}, \ldots, p_s \rangle$, $C_3 = \langle p_{s+1}, \ldots, p_t \rangle$, and two segments $p_r p_{r+1}$ and $p_s p_{s+1}$, where all vertices in C_1 and C_3 are convex vertices of P, while all vertices in C_2 are reflex; see Fig. 1.*

Lemma 2. *Let S be a 2-convex point set and P be a 2-convex polygonization of S. Let K be a non-trivial pocket and let C_1, C_2, and C_3 be the chains from Lemma 1. Then the following two statements hold for every $i \in \{1,2,3\}$.*

(i) *If the chain C_i contains at least three vertices, then C_i is the boundary of a convex polygon with one edge removed.*
(ii) *The convex hull of C_i is a hole in S.*