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Tight bounds for conflict-free chromatic guarding of orthogonal art galleries

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ABSTRACT

The chromatic art gallery problem asks for the minimum number of "colors" t so that a collection of point guards, each assigned one of the t colors, can see the entire polygon subject to some conditions on the colors visible to each point. In this paper, we explore this problem for orthogonal polygons using *orthogonal visibility*—two points p and q are mutually visible if the smallest axis-aligned rectangle containing them lies within the polygon. Our main result establishes that for a *conflict-free* guarding of an orthogonal n-gon, in which at least one of the colors seen by every point is unique, the number of colors is in the worst case $\Theta(\log\log n)$. By contrast, the best known upper bound for orthogonal polygons under standard (non-orthogonal) visibility is $O(\log n)$ colors. We also show that the number of colors needed for *strong* guarding of simple orthogonal polygons, where all the colors visible to a point are unique, is, again in the worst case, $\Theta(\log n)$. Finally, our techniques also help us establish the first non-trivial lower bound of $\Omega(\log\log n/\log\log\log n)$ for conflict-free guarding under standard visibility. To this end we introduce and utilize a novel discrete combinatorial structure called *multicolor tableau*.

1. Introduction

The classic Art Gallery Problem (AGP) posed by Klee in 1973 asks for the minimum number of guards sufficient to watch any art gallery modeled by an n-sided simple polygon P. A guard sees a point in P if the connecting line segment is contained in P. Therefore, a guard watches a star polygon contained in P and the question is to cover P by a collection of stars with smallest possible cardinality. The answer is $\lfloor \frac{n}{3} \rfloor$ as shown by Chvátal [4]. This result was the starting point for a rich body of research about algorithms, complexity and combinatorial aspects for many variants of the original question. Surveys including historical aspects can be found in the seminal monograph by O'Rourke [11], in Shermer [13], and Urrutia [16].

Graph coloring arguments have been frequently used for proving worst case combinatorial bounds for art gallery type questions starting with Fisk's proof [6]. Somehow surprisingly, chromatic versions of the AGP have been proposed and studied only recently. There are two chromatic variants: strong chromatic guarding and conflict-free guarding of a polygon P.

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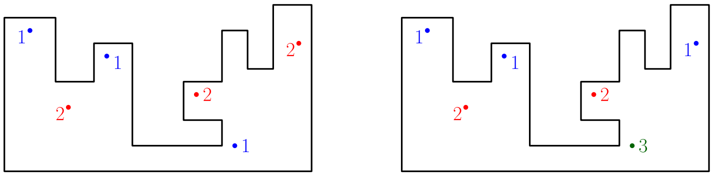

Fig. 1. Example of conflict-free (left) and strong chromatic (right) r-guarding.

In both versions we look for a guard set G and give each guard one of t colors. The chromatic guarding is said to be strong if for each point $p \in P$ all guards $G(p)$ that see p have pairwise different colors [5]. It is conflict-free if in each $G(p)$ there is at least one guard with a unique color, see [1]. The goal is to determine guard sets such that the number of colors sufficient for these purposes is minimal. Observe, in both versions minimizing the number of guards is not part of the objective function. Fig. 1 shows a simple orthogonal polygon with both conflict-free and strong chromatic guardings in the orthogonal visibility model. To grasp the nature of the problem, observe that it has two conflicting aspects. We have to guard the polygon but at the same time we want the guards to hide from each other, since then we can give them the same color. Moreover, we will see a strong dependence of the results on the underlying visibility model, standard vs. orthogonal visibility. We refer to standard and orthogonal visibility as l-visibility (line visibility) and r-visibility, respectively. We use superscripts l and r in the bounds to indicate the model.

Let $\chi^l_{st}(n)$ and $\chi^l_{cf}(n)$ denote the minimal number of colors sufficient for any simple polygon on n vertices in the strong chromatic and in the conflict-free version if based on line visibility.

Here is a short summary of known bounds. For simple orthogonal polygons on n vertices $\chi^l_{cf}(n) \in O(\log n)$, as shown in [1]. The same bound applies to simple general polygons, see [2]. Both proofs are based on subdividing the polygon into weak visibility subpolygons that are in a certain sense independent with respect to conflict-free chromatic guarding. For the strong chromatic version we have $\chi^l_{st}(n) \in \Theta(n)$ for simple polygons and $\chi^l_{st}(n) \in \Omega(\sqrt{n})$ even for the monotone orthogonal case, see [5]. NP-hardness is discussed in [7]. In [5], simple $O(1)$ upper bounds are shown for special polygon classes like spiral polygons and orthogonal staircase polygons combined with line visibility.

Next we state our main contributions for simple orthogonal polygons:

1. For the strong chromatic version we show $\chi^r_{st}(n) \in \Theta(\log n)$.
2. For the conflict-free chromatic version we show $\chi^r_{cf}(n) \in \Theta(\log\log n)$.
3. For line visibility guards we have: $\chi^l_{cf}(n) \in \Omega(\log\log n/\log\log\log n)$.
   This is the first super-constant lower bound also for general simple polygons.

The chromatic AGP versions can be easily interpreted as coloring questions for concrete geometric hypergraphs. Smorodinsky ([15]) gives a nice survey of both practical and theoretical aspects of hypergraph coloring. A special role play hypergraphs that arise in geometry. For example, given a set of points P in the plane and a set of regions $\mathcal{R}$ (e.g. rectangles, disks), we can define the hypergraph $H_{\mathcal{R}}(P) = (P, \{P \cap S | S \in \mathcal{R}\})$. The discrete interval hypergraph $H_{\mathcal{I}}$ is a concrete example of such a hypergraph: We take n points on a line and all possible intervals as regions. It is not difficult to see that $\chi_{cf}(H_{\mathcal{I}}) \in \Theta(\log n)$. As to our AGP versions, we can associate with a given polygon and a guard set a geometric hypergraph. Its vertices are the guards and a hyperedge is defined by a set of guards for which there exists a point that can see exactly these guards. Then one wants to color this hypergraph in a conflict-free or in a strong manner. Another example is the following rectangle hypergraph. The vertex set is a set of n axis-aligned rectangles and each maximal subset of rectangles with a common intersection forms a hyperedge. Here the order for the conflict-free chromatic number is $\Omega(\log n)$ and $O(\log^2 n)$ as shown in [12,15].

Looking at our results, it is not a big surprise that the combination of orthogonal polygons with r-visibility yields the strongest results. This is simply due to additional structural properties and this phenomenon has already been observed for the original AGP. For example, the $\lfloor \frac{n}{4} \rfloor$ tight worst case bound for covering simple orthogonal polygons with general stars can also be proven for r-stars (see [11]) and it holds even for orthogonal polygons with holes, see [8]. Further, while minimizing the number of guards is NP-hard both for simple general and orthogonal polygons if based on line visibility, it becomes polynomially solvable for r-visibility in the simple orthogonal case, see [10,17]. The latter result is based on the solution of the strong perfect graph conjecture.

The paper is organized as follows. We give necessary basic definitions in the next section. Then we prove upper bounds in Section 3 using techniques developed in [1,2]. That means we also subdivide a simple orthogonal polygon into histograms which are independent with respect to chromatic guarding. To deal with a single histogram we introduce the notion of its *spine tree*. The spine tree provides an elegant and efficient way to describe r-visibility properties of the histogram. Our main contributions are the lower bound proofs in Section 4. Especially, we introduce a novel combinatorial structure called multicolor tableau. This structure enables us to show a first super-constant lower bound for chromatic conflict-free guarding based on the line visibility model.