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Exact Minkowski sums of polygons with holes ☆

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ABSTRACT

We present an efficient algorithm that computes the Minkowski sum of two polygons, which may have holes. The new algorithm is based on the convolution approach. Its efficiency stems in part from a property for Minkowski sums of polygons with holes: Given two polygons with holes, for each input polygon we can fill up the holes that are relatively small compared to the other polygon. Specifically, we can always fill up all the holes of at least one polygon, transforming it into a polygon without holes, and still obtain exactly the same Minkowski sum. Obliterating holes in the input summands speeds up the computation of Minkowski sums.
We introduce a robust implementation of the new algorithm, which follows the Exact Geometric Computation paradigm and thus guarantees exact results. We also present an empirical comparison of the performance of Minkowski sum construction of various input examples, where we show that the implementation of the new algorithm exhibits better performance than several other implementations in many cases.
The software is available as part of the *2D Minkowski Sums* package of CGAL (Computational Geometry Algorithms Library), starting from Release 4.7. Additional information and supplementary material is available at our project page http://acg.cs.tau.ac.il/projects/rc.

1. Introduction

Let P and Q be two point sets in $\mathbb{R}^d$. The Minkowski sum of P and Q is defined as $P \oplus Q = \{p + q \mid p \in P, q \in Q\}$. In this paper we focus on the computation of Minkowski sums of polygons in the plane, that is, polygons which may have holes. Minkowski sums are ubiquitous in many fields and applications including robot motion planning [1], assembly planning [2], computer-aided design [3], and collision detection in general [4].

Let m and n denote the number of vertices of P and Q, respectively. The boundary of the Minkowski sum $P \oplus Q$, is formed from the sum of pairs of edges and vertices, where each summand is from a different polygon. Since there are $O(mn)$ such pairs, the sum induces an arrangement of $O(m^2n^2)$ edges. This bound is tight in the worst case. If both P and Q are convex, the Minkowski sum has at most $m + n$ vertices and may be computed efficiently in $O(m + n)$ time. If only one of the polygons is convex, the sum has $O(mn)$ vertices and may be computed in $O(mn \log(mn))$ time; see, e.g., [5].

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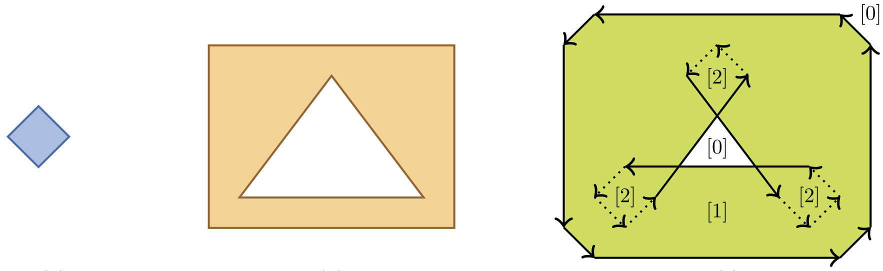

(a) (b) (c)

Fig. 1. The convolution (c) of a convex polygon (a) and a non-convex polygon (b), which has a self-intersecting cycle, drawn as a chain of directed line segments (some of which are dotted). The winding numbers of the faces of the arrangement induced by the segments forming the cycle are indicated in brackets. The Minkowski sum of the two polygons is shaded. The dotted edges are filtered out by the reduced convolution method.

1.1. Terminology and related work

During the last four decades many algorithms to compute the Minkowski sum of polygons or polyhedra were introduced. For exact two-dimensional solutions see, e.g., [5]. For approximate solutions see, e.g., [6] and [7]. For exact and approximate three-dimensional solutions see, e.g., [8], [9], [10], and [11].

In this paper, we primarily deal with *polygons with holes*.

Definition 1 *(Polygon with holes).* A polygon with holes is a connected and closed point set in the plane. It is bounded by one closed non-crossing polygonal chain, which constitutes the outer boundary and zero or more closed non-self-intersecting polygonal chains, each constituting an inner boundary. Each inner boundary bounds a connected and open point set referred to as a hole. Holes of a polygon with holes are pairwise disjoint and do not intersect with the outer boundary.

Observe that while the outer boundary and inner boundaries of a polygon with holes cannot cross each other, they can still intersect (touch) each other. The outer boundary, being part of the polygon with holes, can touch itself. On the other hand, an inner boundary is not part of the polygon, and thus cannot touch itself. (When two inner boundaries touch, they simply form two separated holes.)

Theorem 1, Theorem 2, and Theorem 3 and their proofs (see Section 2) use the above definition without exploiting the fact that the chains are made of line segments; that is, neither the theorems nor the proofs rely on the fact that the boundaries are piecewise linear. However, our algorithms that compute the Minkowski sum of two polygons (regardless of whether they have holes or not) accept (linear) polygons as input and produce (linear) polygons as output.

Computing the Minkowski sum of two convex polygons P and Q is rather easy. As $P \oplus Q$ is a convex polygon bounded by copies of the edges of P and Q ordered according to their slopes, the Minkowski sum can be computed using an operation similar to merging two sorted lists of numbers. If the polygons are not convex, it is possible to use one of the two general approaches listed below.

Decomposition Algorithms that follow the decomposition approach decompose P and Q into two sets of convex sub-polygons. Then, they compute the pairwise sums using the simple procedure described above. Finally, they compute the union of the pairwise sums. This approach was first proposed by Lozano-Pérez [12]. The performance of this approach heavily depends on the method that computes the convex decomposition of the input polygons. Agarwal et al. [13] described an implementation of the first exact and robust version of the decomposition approach, which handles degeneracies. They also tested different decomposition methods, but none of them handles polygons with holes.

Ghosh [14] introduced *slope diagrams*—a data structure that was used later on by some of us to construct Minkowski sums of bounded convex polyhedra in 3D [15]. Hachenberger [8] constructed Minkowski sums of general polyhedra in 3D. Both implementations are based on the Computational Geometry Algorithms Library (CGAL) [16] and follow the Exact Geometric Computation (EGC) paradigm.

Convolution The interior of the polygon lies to the left of every edge of the polygon. It implies that vertices along the outer boundary of a polygon wind in counterclockwise order around the interior of the polygon, and vertices along an inner boundary (the boundary of a hole) wind in clockwise order around the hole. Let $(p_0, \ldots, p_{k-1})$ and $(q_0, \ldots, q_{\ell-1})$ denote the sequence of vertices along one boundary component of each of the input polygons P and Q, respectively. The *convolution* of these two polygons, denoted $P * Q$, is a collection of line segments of the forms[1] $[p_i + q_j, p_{i+1} + q_j]$ and $[p_i + q_j, p_i + q_{j+1}]$, repeating this for every pair of boundary components one of P and one of Q. The former form appears

[1] Incrementing and decrementing indices of vertices on some boundary is carried out modulo the number of vertices on that boundary.

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