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Reconstruction of the path graph

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A B S T R A C T

Let P be a set of $n \geq 5$ points in convex position in the plane. The path graph $G(P)$ of P is an abstract graph whose vertices are non-crossing spanning paths of P, such that two paths are adjacent if one can be obtained from the other by deleting an edge and adding another edge.

We prove that the automorphism group of $G(P)$ is isomorphic to D_n, the dihedral group of order $2n$. The heart of the proof is an algorithm that first identifies the vertices of $G(P)$ that correspond to boundary paths of P, where the identification is unique up to an automorphism of $K(P)$ as a geometric graph, and then identifies (uniquely) all edges of each path represented by a vertex of $G(P)$. The complexity of the algorithm is $O(N \log N)$ where N is the number of vertices of $G(P)$.

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1. Introduction

A *geometric graph* is a graph whose vertices are a finite set of points in general position in the plane, and whose edges are closed segments connecting distinct points. We consider the complete convex geometric graph $K(P)$, in which the vertex set is a convex set P of n points in the plane, and the edges are all segments connecting pairs of vertices. Without loss of generality we will henceforth assume that P is the vertex set of a regular n-gon.

Definition 1. Let P be a set of n points in the plane. The *path graph* $G(P)$ is defined as follows. The vertices of $G(P)$ are the simple (i.e., non-crossing) spanning paths of $K(P)$. Two such vertices are adjacent in $G(P)$ if they differ in exactly two edges, i.e., if one can be obtained from the other by deleting an edge and adding another edge.

The path graph was introduced in 2001 by Rivera-Campo and Urrutia-Galicia [13] who showed that when P is in convex position, $G(P)$ is Hamiltonian. Following [13], several works studied $G(P)$ in the convex case. Akl et al. [3] showed that $|V(G(P))| = n2^{n-3}$ and that $\mathrm{diam}(G(P)) \leq 2n-5$. Chang and Wu [6] determined the diameter exactly, showing that $\mathrm{diam}(G(P)) = 2n-5$ for $n = 3, 4$ and $\mathrm{diam}(G(P)) = 2n-6$ for $n \geq 5$. Fabila-Monroy et al. [8] showed that the chromatic

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number of $G(P)$ is n. Wu et al. [15] presented algorithms for generating plane spanning paths efficiently. The general (i.e., non-convex) case is less-studied, and it is not known even whether $G(P)$ is connected for all P (see [3]).

The study of $G(P)$ evolved from the study of the *geometric tree graph* $\mathcal{T}(P)$ which has all non-crossing spanning trees of P as its vertices, and two vertices are adjacent in $G(P)$ if they differ in exactly two edges. Defined by Avis and Fukuda [4] as the geometric counterpart of the classical *tree graph* [7], $\mathcal{T}(P)$ was studied in quite a few works, both in the convex and in the general case (e.g., [1,2,9–12]).

Some of the central results on $\mathcal{T}(P)$, such as Hamiltonicity and upper/lower bounds on the diameter (see [4,10]) already have counterparts for $G(P)$ (proved in [3,6,13]). In this paper we establish a counterpart of another result: exact determination of the *automorphism group* in the convex case. For $\mathcal{T}(P)$, Hernando et al. [10] showed that $\mathrm{Aut}(\mathcal{T}(P))$ is D_n, the dihedral group of rotations and reflections of a regular n-gon. Since $\mathrm{Aut}(K(P)) \cong D_n$, it follows that D_n is isomorphic to a subgroup of $\mathrm{Aut}(G(P))$.

Our main result is that there are no other automorphisms on $G(P)$.

Theorem 2. *Let P be a set of $n \geq 5$ points in convex position in the plane, and let $G(P)$ be its path graph. Then* $\mathrm{Aut}(G(P)) \cong D_n$.

The proof of Theorem 2 relies on an algorithm that allows recovering all edges of each path represented by a vertex of $G(P)$ (up to an automorphism of $K(P)$ as a geometric graph), given $G(P)$ as an abstract graph. The algorithm exploits analysis of maximal cliques in $G(P)$, following an approach pioneered by Urrutia-Galicia [14]. First, we use the structure of the max-cliques to identify an ordered subset of n vertices of $G(P)$ that corresponds to the boundary paths of P, where the identification is fixed up to an automorphism of $K(P)$. Then we show that once the ordered subset is fixed, all edges of each path can be determined uniquely by examining distances between various vertices of $G(P)$. The running time of the algorithm is $O(N \log N)$ where $N = |V(G(P))|$, which is close to optimal, since for each of the N vertices of $G(P)$ we recover the $n-1 = \Theta(\log N)$ edges in the path it represents. It should be noted that the determination of $\mathrm{Aut}(\mathcal{T}(P))$ in [10] is non-constructive, and no efficient algorithm is known for full recovery of $\mathcal{T}(P)$. In this sense, our result is stronger than the analogous result on $\mathcal{T}(P)$. Likewise, while the technique of Urrutia-Galicia [14] was used in several previous works, this is the first time it is used for complete recovery of $G(P)$, thus solving completely a natural *graph reconstruction* problem (see, e.g., [5] for a definition and survey of reconstruction problems).

The paper is organized as follows. Hereinafter, we present notations and a simple observation used throughout the paper. In Section 2 we study the structure of maximal cliques in $G(P)$. In Section 3 we prove the main theorem. We conclude the paper with a complexity analysis, in Section 4, and a few open problems.

Notations

In this section we present notations and simple observations that will be used in the sequel.

Throughout the paper, P is a set of points in convex position in the plane. The edges of $K(P)$, the complete geometric graph on P, are divided into two classes: n *boundary edges* of $\mathrm{Conv}(P)$ and $\binom{n}{2} - n$ *diagonals*, i.e., edges internal to $\mathrm{Conv}(P)$. We denote the set of boundary edges by $\mathcal{B}(P)$, and say that $x, y \in P$ are *neighboring* if $(x, y) \in \mathcal{B}(P)$. An automorphism of $K(P)$ as a geometric graph is an automorphism of $K(P)$ as an abstract graph that, in addition, maps crossing edges into crossing edges and non-crossing edges into non-crossing edges.

As defined above, $G(P)$ denotes the (non-crossing) spanning path graph of P. For $v \in V(G(P))$, $P(v)$ denotes the path represented by v. For the sake of convenience, we sometimes use the term $P(v)$ also for the edge-set of the path represented by v. We stress that we usually denote this edge-set by v; the notation $P(v)$ is used for it only in places when the meaning is clear from the context.

The set of boundary edges of $P(v)$, that is, $P(v) \cap \mathcal{B}(P)$, is denoted by $\mathcal{B}(v)$. The set of diagonals of $P(v)$ is denoted by $\mathcal{D}(v) = P(v) \setminus \mathcal{B}(v)$. $P(v)$ is called a *boundary path* if all its edges are boundary edges. We denote the set of vertices of $G(P)$ that represent boundary paths by $\mathcal{B}$. Note that while $\mathcal{B}(v)$ denotes the boundary edges of a specific path, $\mathcal{B}$ denotes a subset of the vertices of $G(P)$.

For any graph G, the distance between vertices x, y, denoted $\mathrm{dist}(x, y)$, is the shortest length of a path in G from x to y. The distance of a vertex from a set $\mathcal{C}$ of vertices is defined as $\mathrm{dist}(x, \mathcal{C}) = \min_{y \in \mathcal{C}} \mathrm{dist}(x, y)$. The *degree* of a vertex v in a graph G is the number of edges of G that emanate from v, and is denoted by $\deg_G(v)$. A vertex is called a *leaf* if its degree is 1. An edge is called a *leaf edge* if one of its endpoints is a leaf. A vertex that is not a leaf is called an *internal vertex*.

We use the following simple observation on the structure of simple spanning paths of P.

Observation 3. *Let S be a simple spanning path of a set P of points in convex position in the plane. Then:*

1. *Both leaf edges of S are boundary edges.*
2. *If S is not a boundary path, then its leaves cannot be neighboring vertices of the boundary.*

The easy proof of the observation is omitted.

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