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Visibility representations of boxes in 2.5 dimensions ☆

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A B S T R A C T

Visibility representations are a well-known paradigm to represent graphs. From a high-level perspective, a visibility representation of a graph G maps the vertices of G to non-overlapping geometric objects and the edges of G to visibilities, i.e., segments that do not intersect any geometric object other than at their end-points. In this paper, we initiate the study of 2.5D box visibility representations (2.5D-BRs) where vertices are mapped to 3D boxes having the bottom face in the plane $z = 0$ and edges are unobstructed lines of sight parallel to the x- or y-axis. We prove that: (i) Every complete bipartite graph admits a 2.5D-BR; (ii) The complete graph K_n admits a 2.5D-BR if and only if $n \leqslant 19$; (iii) Every graph with pathwidth at most 7 admits a 2.5D-BR, which can be computed in linear time. We then turn our attention to 2.5D grid box representations (2.5D-GBRs), which are 2.5D-BRs such that the bottom face of every box is a unit square at integer coordinates. We show that an n-vertex graph that admits a 2.5D-GBR has at most $4n - 6\sqrt{n}$ edges and this bound is tight. Finally, we prove that deciding whether a given graph G admits a 2.5D-GBR with a given footprint is NP-complete (the footprint of a 2.5D-BR Γ is the set of bottom faces of the boxes in Γ).

1. Introduction

A *visibility representation* (VR) of a graph G maps the vertices of G to non-overlapping geometric objects and the edges of G to *visibilities*, i.e., segments that do not intersect any geometric object other than at their end-points. Depending on the type of geometric objects representing the vertices and on the rules used for the visibilities, different types of representations have been studied in computational geometry and graph drawing.

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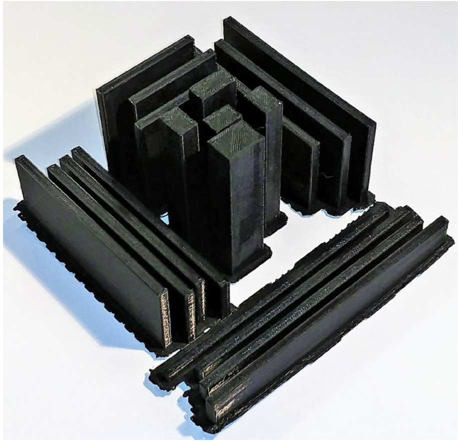

Fig. 1. A 3D printed model of a 2.5D-BR of K_{19} described in the proof of Lemma 7.

A *bar visibility representation* (BVR) maps the vertices to horizontal segments, called *bars*, while visibilities are vertical segments. BVRs were introduced in the 80s as a modeling tool for VLSI problems [16,28,29,34,36,37]. The graphs that admit a BVR are planar and they have been characterized under various models [16,29,34,37].

Extensions and generalizations of BVRs have been proposed in order to enlarge the family of representable graphs. In a *rectangle visibility representation* (*RVR*) the vertices are axis-aligned rectangles, while visibilities are both horizontal or vertical segments [4,5,10,12,13,24,30,32]. RVRs can exist only for graphs with thickness at most 2 and with at most $6n - 20$ edges [24]. Recognizing these graphs is NP-hard in general [30] and can be done in polynomial time in some restricted cases [4,32]. Generalizations of RVRs where orthogonal shapes other than rectangles are used to represent the vertices have been recently proposed [15,27]. Another generalization of BVRs are *bar k-visibility representations* (*k-BVRs*), where each visibility segment can "see" through at most k bars. Dean et al. [11] proved that the graphs admitting a 1-BVR have at most $6n - 20$ edges. Felsner and Massow [20] showed that there exist graphs with a 1-BVR whose thickness is 3. The relationship between 1-BVRs and 1-planar graphs has also been investigated [1,7,17,33].

RVRs are extended to 3D space by *Z-parallel Visibility Representations* (*ZPRs*), where vertices are axis-aligned rectangles belonging to planes parallel to the xy-plane, while visibilities are parallel to the z-axis. Bose et al. [6] proved that K_{22} admits a ZPR, while K_{56} does not. Štola [31] subsequently reduced the upper bound on the size of the largest representable complete graph by showing that K_{51} does not admits a ZPR. Fekete et al. [18] showed that K_7 is the largest complete graph that admits a ZPR if unit squares are used to represent the vertices. A different extension of RVRs to 3D space are the *box visibility representations* (*BRs*) where vertices are 3D boxes, while visibilities are parallel to the x-, y- and z- axis. This model was studied by Fekete and Meijer [19] who proved that K_{56} admits a BR, while K_{184} does not.

In this paper we introduce *2.5D box visibility representations* (*2.5D-BRs*) where vertices are 3D boxes whose bottom faces lie in the plane $z = 0$ and visibilities are parallel to the x- and y-axis. Like the other 3D models that use the third dimension, 2.5D-BRs overcome some limitations of the 2D models. For example, graphs with arbitrary thickness can be realized. In addition 2.5D-BRs seem to be simpler than other 3D models from a visual complexity point of view and have the advantage that they can be physically realized, for example by 3D printers or by using physical boxes, since no box "flies" in the air but they all stand on a common plane. (Fig. 1 shows a 3D printed version of the 2.5D-BR of K_{19} used to prove Lemma 7.) Furthermore, in the context of GNSS applications in urban areas, Carmi et al. [9] studied graphs defined by visibilities between buildings. Our study can be regarded as an investigation of the properties of a subclass of such visibility graphs in which visibilities are restricted to be parallel to the x- or y-axis.

- We show that every complete bipartite graph admits a 2.5D-BR and that the complete graph K_n admits a 2.5D-BR if and only if $n \leqslant 19$ (Section 3). These results have several implications: (i) there exist graphs that admit a 2.5D-BR and have arbitrary thickness; (ii) the maximum density of a graph that admits a 2.5D-BR is in the range $[\frac{1}{4}, \frac{9}{19}]$; (iii) the family of graphs that admit a 2.5D-BR is not closed under edge contraction.
- We show that every graph with pathwidth at most 7 admits a 2.5D-BR. In particular we describe a linear-time construction for this family of graphs (Section 4). We recall that the pathwidth is an upper bound for the treewidth. Visibility representations of graphs with bounded treewidth have been studied by Bose et al. [5], who showed that every graph with treewidth at most 4 admits an RVR (and thus a 2.5D-BR).
- We then study *2.5D grid box representations* (*2.5D-GBRs*) which are 2.5D-BRs such that the bottom face of every box is a unit square with corners at integer coordinates (Section 5). We show that an n-vertex graph that admits a 2.5D-GBR has at most $4n - 6\sqrt{n}$ edges and that this bound is tight. It is worth remarking that papers about VRs often concentrate on settings with restrictions on the size of the objects representing the vertices. Examples of these models are unit bar VRs [14], unit square VRs [10], and unit box VRs [19].